

THE LIMIT-POINT AND LIMIT-CIRCLE CASES  
OF SECOND AND FOURTH ORDER  
DIFFERENTIAL EQUATIONS

By

DAVID HARTMAN TAYLOR

"

Bachelor of Science  
Southwestern Oklahoma State University  
Weatherford, Oklahoma  
1966

Master of Arts  
University of Missouri-Columbia  
Columbia, Missouri  
1968

Submitted to the Faculty of the Graduate College  
of the Oklahoma State University  
in partial fulfillment of the requirements  
for the Degree of  
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Thesis Approved:

*Marvin S. Keener*

Thesis Adviser

*John J. Jaber*

*Chai Alwad*

*IS Alchelle*

*James Thomas Burnham*

*N N Durham*

Dean of the Graduate College

939008

## PREFACE

This study is concerned with some of the problems that arise when considering expansions of arbitrary functions in terms of eigenfunctions associated with formally self-adjoint differential equations. The primary objective is to consider necessary and sufficient conditions that place a given differential expression into a particular limit-p case.

I am deeply indebted to my thesis adviser, Dr. Marvin Keener. Without his aid, this thesis could never have been completed. I also wish to thank the other members of my committee, Professor John Jobe, Professor Shair Ahmad, Professor Doug Aichele, and Professor James Burnham for their assistance. To Dr. Jobe, I owe the deepest debt for teaching me to read and write mathematics.

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## TABLE OF CONTENTS

Chapter	Page
I. INTRODUCTION . . . . .	1
II. THE SECOND-ORDER CASE . . . . .	17
III. THE FOURTH-ORDER CASE . . . . .	22
IV. EIGENFUNCTION EXPANSIONS . . . . .	44
V. THE DEFICIENCY INDEX PROBLEM . . . . .	67
VI. SUMMARY . . . . .	108
BIBLIOGRAPHY . . . . .	113

## CHAPTER I

### INTRODUCTION

This paper is concerned with some aspects of a broad area in differential equations widely known as the eigenvalue problem. The problem in its classical form consists of a linear differential expression  $Ly$  defined on a compact interval of the real line together with a set of boundary conditions at the endpoints of the interval. The desired result is to determine a sequence of so-called eigenfunctions associated with the expression  $Ly$  and then, under certain conditions, to represent a function  $f$  as an infinite series in these eigenfunctions. The problem described above may perhaps be made clearer by the use of a simple example.

EXAMPLE 1.1: Consider the linear differential expression  $Ly$  given by

$$Ly = -y'' \tag{1.1}$$

defined on the interval  $0 \leq x \leq \pi$  with the boundary conditions

$$y(0) = y(\pi) = 0, \tag{1.2}$$

where the primes represent differentiation with respect to  $x$ . An eigenvalue of  $Ly$  is a complex number  $\lambda$  for which there exists a nontrivial function  $y(x, \lambda)$ , called the eigenfunction corresponding to  $\lambda$ , defined on the interval  $[0, \pi]$  such that  $y'$  is absolutely continuous on that interval and  $y$  satisfies the differential equation  $Ly = \lambda y$  with boundary

conditions (1.2). It is easily verified that the only values of  $\lambda$  for which a nontrivial function  $y(x, \lambda)$  exists are those real numbers  $\lambda = n^2$ ,  $n = 1, 2, \dots$ . Thus the eigenvalues associated with  $Ly$  are  $\lambda_n = n^2$ ,  $n = 1, 2, \dots$ , and the corresponding eigenfunctions are  $y(x, n^2) = \sin nx$ ,  $n = 1, 2, \dots$ . If  $f$  is an absolutely continuous function defined on the interval  $[0, \pi]$  that vanishes at the endpoints of the interval, then  $f$  may be represented by the infinite series

$$(2/\pi) \sum_{n=1}^{\infty} \left( \int_0^{\pi} \sin(nt) f(t) dt \right) \sin(nx). \quad (1.3)$$

The term "represents" can mean various things. For example, one common interpretation is that the series (1.3) represents  $f$  on the interval if for  $S_n(x)$  the  $n$ -th partial sum,

$$\lim_{n \rightarrow \infty} \int_0^{\pi} |S_n(x) - f(x)|^2 dx = 0.$$

That is, the series converges in the mean to  $f$  (or converges to  $f$  in the norm of the Hilbert space  $L^2(0, \pi)$ ). Another common interpretation is that the series represents  $f$  in the sense

$$\lim_{n \rightarrow \infty} S_n(x) = f(x)$$

for each  $x$  in the interval. This is commonly called direct convergence. The latter form of convergence will be considered for even order differential expressions.

The series (1.3) is the well known Fourier sine series. The subject of Fourier series representation of functions is a part of a more general topic called Sturm-Liouville theory. Interest in Fourier series expansion of functions has existed almost as long as calculus.

The topics in mathematical physics of interest of that period included boundary value problems in vibration of strings stretched between fixed points and vibrations of bars or columns of air, associated with mathematical theories of musical vibrations. Early contributors to the theory of vibrating strings were Brook Taylor, Daniel Bernoulli, L. Euler, and d'Alembert. By the 1750's, the latter three mathematicians had advanced the theory to the stage that the partial differential equation  $y_{tt} = a^2 y_{xx}$  was known and a solution of the boundary value problem had been found. Also, the concept of fundamental nodes of vibration led those men to notions of superposition of solutions, that is, to a solution of the form

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \cos(n\pi at)$$

and thus to the matter of representing an arbitrary function by a trigonometric series. In 1822, J. B. Fourier presented many instructive examples of expansions in trigonometric series in connection with boundary value problems in the conduction of heat. In 1829, P. Dirichlet established general conditions on a function sufficient to ensure the convergence of its Fourier series to values of the function.

In 1830, J. C. F. Sturm and J. Liouville almost simultaneously developed a systematic theory of the expansion of arbitrary functions in eigenfunctions associated with the formally self-adjoint differential expression  $Ly = -(py')' + qy$  on a compact interval with  $p$  and  $q$  real-valued and  $p > 0$ . (These last terms will be described shortly.) This is the natural extension of the theory of Fourier which has its base in the differential expression (1.1) on a compact interval. In 1910,



Hermann Weyl published a major paper that generalized the theory of Sturm and Liouville to the singular differential expression given by  $Ly = -(py')' + qy$  defined on the half-line  $0 \leq x < \infty$  where the real-valued functions  $p$  and  $q$  have finite limits as  $x$  tends to zero from the right. It was in this generalization that he introduced some simple geometric concepts in order to determine whether or not the function to be expanded must first satisfy a boundary condition at  $x = \infty$ .

The subject of considering Weyl's generalization of the Sturm-Liouville theory (hereafter called Weyl's theory) was essentially stagnant for approximately thirty-five years following Weyl's work. There were, however, two attempts made during this period to further generalize Weyl's theory to differential expressions of higher orders. The first of these was undertaken by W. Windau [81] in 1921 to generalize the problem to fourth-order expressions. The second attempt was made by D. Sin in 1939-40 for arbitrary order. However, both authors made essential errors in the beginnings of their developments and as a result, failed to succeed in satisfactorily extending Weyl's theory to higher order problems. Glazman [46], in 1950, confirmed that these authors were in error. Weyl's theory was picked up successfully in the mid 1940's by E. C. Titchmarsh [74] and was further developed and modernized using methods differing from those employed earlier. Weyl had used the theory of integral equations and Stone [71] presented an alternative method by proceeding via the general theory of linear operators in Hilbert space. Titchmarsh sought to avoid both these methods and proceeded by means of contour integrations and the calculus of residues. He notes that many times, however, he is doing no more than adopting the general theory of linear operators to the particular

case being considered.

K. Kodaira [59], in 1950, published a correct generalization of Weyl's theory to differential expressions of any even order. He proceeded by means of general linear operator theory. In 1955, W. N. Everitt, in his Doctor of Philosophy thesis (Oxford), used the methods of Titchmarsh together with some ideas of Kodaira and generalized Weyl's theory to differential expressions of the fourth order. Kodaira's results require a strong background in operator theory to read, while Everitt's [30] analysis is based on elementary methods of complex function theory. In this sense, Everitt's analysis may be considered to be more elementary. Everitt published his results in 1963.

From this general problem, there has emerged a problem of determining a classification of linear, formally self-adjoint differential expressions of order  $2n$  into so-called limit- $p$  cases,  $n \leq p \leq 2n$ . The consideration of this classification stems from the necessity of determining whether or not boundary conditions at the singular end of the interval must be imposed upon the function in order that it may be expanded in terms of the eigenfunctions. As will be seen, it turns out that if the differential expression is in the limit- $n$  case, then no boundary conditions at the singular endpoint need be imposed upon the functions to be expanded. In each of the other cases, a boundary condition must be imposed. The classification limit- $n$  is usually called limit-point and that of limit- $2n$  is usually called limit-circle. This terminology has come from the second-order case and has an elegant geometric interpretation. The problem of classifying these differential expressions is a fairly recent one although Weyl did give some criteria

involving the growth of the coefficients that place the differential expression into the various limit- $p$  cases. The majority of published results have appeared since 1965.

An interesting problem that precisely parallels that of classification is one of determining the maximum number (up to linear independence) of solutions to the problem

$$Ly = \lambda y, \quad 0 \leq x < \infty, \quad \text{Im } \lambda \neq 0 \quad (1.4)$$

that lie in the Hilbert space  $L^2(0, \infty)$ . It is the case that for  $\text{Im } \lambda \neq 0$ , but otherwise an arbitrary complex number, the  $2n$ -th order differential expression  $Ly$  is in the limit- $p$  case,  $n \leq p \leq 2n$ , if and only if there exists a basis for the solution space of (1.4) that contains precisely  $p$  functions in  $L^2(0, \infty)$  and no basis contains more than  $p$  such solutions. The equivalence of these two problems will be discussed in Chapter V.

The procedure in this paper will be to first consider the second-order case by generalizing the results of Titchmarsh [74] to the concepts of quasi-derivatives. These results are presented in order to have a compact presentation of those results first obtained by Weyl. It is also the case for second-order expressions that the geometry involved is quite elegant. The theory will then be extended to the  $2n$ -th order case via the fourth-order case. The generalization from fourth to  $2n$ -th order is direct while the generalization from second order is not, as witnessed by Windau and Sin.

One of the primary goals is consideration of the problem of classification of second and fourth-order differential expressions into the various limit- $p$  cases. Much is known in the second-order case, while

less has been accomplished in the fourth-order case. There is only one known method for obtaining results on the classification of  $2n$ -th order expressions and this method will be illustrated for the fourth-order case.

Before proceeding, some background material and notations will be established. Let  $(a,b)$  be an interval in  $\mathbb{R}$  and let  $A: (a,b) \rightarrow M^n$  and  $y: (a,b) \rightarrow \mathbb{C}^n$  be functions where  $M^n$  is the set of  $n \times n$  complex matrices and  $\mathbb{C}^n$  is the set of complex  $n$ -vectors. The usual conventions as to the meaning of  $dy/dx$  and  $dA/dx$  as used in differential equations will be adopted. Also, let  $\|A\|$  and  $\|y\|$  denote a suitable norm on these functions. A discussion of this notation may be found in the book by Struble [72]. The notations  $\bar{z}$  for the complex conjugate of the function (matrix, number)  $z$ ,  $A^*$  for the transposed conjugate of the matrix  $A$  and  $A^T$  for the transpose of  $A$  will be adopted. The following criterion of the meaning of a function being a solution of a differential equation will be assumed. The prime denotes differentiation with respect to  $x$ .

**DEFINITION 1.1:** A vector function  $y(x)$  is said to be a solution of the matrix differential equation  $y'(x) = A(x)y(x)$ ,  $a < x < b$ , if and only if  $y$  is absolutely continuous in every compact subinterval of  $(a,b)$  and satisfies the differential equation almost everywhere in  $(a,b)$ . A vector or matrix function is said to be absolutely continuous in a compact interval if each component of the vector or matrix is absolutely continuous in that interval.

The following definition of analyticity is standard.

**DEFINITION 1.2:** Let  $F: D \rightarrow \mathbb{C}^k$ ,  $D$  a domain in  $\mathbb{C}^n$ . Then  $F$  is said to be analytic at a point  $w_0$  in  $D$  if and only if in some neighborhood

$\|w - w_0\| < \delta$ ,  $\delta > 0$ , of  $w_0$  in  $D$ , each component  $F_j$  of  $F$  is continuous in  $w$  and is analytic in each component  $w_k$  of  $w$  when all other components are held fixed.

Following is a statement of the existence and uniqueness theorem that will be used. This theorem is standard and its proof may be found in many texts on ordinary differential equations, for example, Struble [72], Naimark [65], or Coddington and Levinson [7].

**THEOREM 1.1:** Suppose  $p_0, p_1, \dots$ , and  $p_n$  are real-valued functions such that each  $p_k$  is measurable on an interval  $(a, b)$ ,  $-\infty \leq a < b \leq \infty$  and such that each  $p_k$  is locally integrable on the interval  $(a, b)$  while  $p_0$  is of constant sign on the interval. Let  $C$  be a fixed vector in  $\mathbb{C}^n$ ,  $x_0$  a real number such that  $a < x_0 < b$  and  $\lambda$  a complex parameter. Let the  $2n \times 2n$  matrix  $A$  be given by

$$A = \begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 1 & & \\ & & & \vdots & & \\ & & & & 1/p_0 & \\ & & & & \vdots & \\ & & & & & -1 \\ & & & & & & p_{n-1} & & -1 \\ & & & & & & p_n^{-\lambda} & & 0 \end{bmatrix} \quad (1.5)$$

where the partitioning is between the  $n$ -th and  $(n+1)$ -st columns and rows and unmarked entries are zero. Then there exists a unique vector-valued function  $y(x, C, \lambda)$  that is locally absolutely continuous on

$a < x < b$ , is an entire function of  $\lambda$  for each fixed  $x$ , and satisfies the differential equation  $y' = A(\lambda)y$  with the initial condition  $y(x_0, C, \lambda) = C$ .

Some previously used terminology will now be discussed. Suppose the coefficients  $q_k(x)$ ,  $k = 0, 1, \dots, n$  of the differential expression

$$Ly = q_0 y^{(m)} + q_1 y^{(m-1)} + \dots + q_m y$$

have continuous derivatives up to order  $(m - k)$  inclusive on the open interval  $(a, b)$ . Let  $L^*z$  be the differential expression given by

$$L^*z = (-1)^m (\bar{q}_0 z)^{(m)} + (-1)^{m-1} (\bar{q}_1 z)^{(m-1)} + \dots + \bar{q}_m z$$

called the adjoint of  $L$ . If all the coefficients of a formally self-adjoint differential expression are real-valued, then the expression is necessarily of even order and can be put into the form

$$Ly = (-1)^n (p_0 y^{(n)})^{(n)} + (-1)^{n-1} (p_1 y^{(n-1)})^{(n-1)} + \dots + p_n y. \quad (1.6)$$

This result may be found in Section 1.5 of [65]. The expression  $(u, v)$  will denote the inner product of two functions in  $L^2(a, b)$  defined by

$$(u, v) = \int_a^b u(x) \bar{v}(x) dx.$$

Suppose  $Ly$  is a formally self-adjoint differential expression with real-valued coefficients and define

$$U_j y = \sum_{k=1}^{2n} M_{jk} y^{(k-1)}(a), \quad V_j y = \sum_{k=1}^{2n} N_{jk} y^{(k-1)}(b) \quad (1.7)$$

for  $j = 1, 2, \dots, 2n-1$  where  $M_{jk}$  and  $N_{jk}$  are real constants. Denote the relations  $U_j y + V_j y = 0$ ,  $j = 1, 2, \dots, 2n-1$  by

$$(U + V)y = 0. \quad (1.8)$$

The relations (1.8) are called Sturmian boundary conditions.

DEFINITION 1.3: Let  $\lambda$  be a complex number. Then the problem

$$Ly = \lambda y, (U + V)y = 0 \quad (1.9)$$

is called an eigenvalue problem and is said to be self-adjoint if and only if for each pair of functions  $u$  and  $v$  that has continuous  $n$ -th order derivatives on the interval  $a < x < b$  and satisfies the condition  $(U + V)y = 0$  also satisfies the condition  $(Lu, v) = (u, Lv)$ .

Thus, a formally self-adjoint differential expression is the differential expression associated with a self-adjoint problem (1.9).

Assume that the coefficients  $p_k$  of (1.6) are locally integrable on the interval. Then a more general differential expression is described if the coefficients are not assumed to be differentiable. A differential expression of the form (1.6) is still called formally self-adjoint under these more general conditions and it is convenient to consider a generalization of the derivative called a quasi-derivative. The theory of quasi-derivatives is very similar to, and in some cases simpler than, that of ordinary derivatives and differential equations. The terms "derivative" and "differential equation" will be retained even when it may be more proper to use the expressions "quasi-derivative" and "quasi-differential equation." An excellent account of these concepts may be found in Naimark [65]. Quasi-derivatives are defined below and it should be noted that the definitions are dependent upon the differential equation being considered. The primary difference between differential

equations and quasi-differential equations is that, in the latter case, no requirements as to the differentiability of the coefficient functions are assumed. In the case of the following definition, the differential expression being considered is (1.6).

DEFINITION 1.4: Let  $y$  be a vector function. Then the various quasi-derivatives  $y^{[k]}$ , with respect to (1.6), are defined by

$$y^{[k]} = y^{(k)}, \quad 0 \leq k \leq n-1, \quad y^{[n]} = p_0 y^{(n)},$$

$$y^{[n+k]} = p_k y^{(n-k)} - (y^{[n+k-1]})', \quad 1 \leq k \leq n,$$

where  $y^{(k)}$  denotes the  $k$ -th order ordinary derivative.

The connections among the  $2n$ -th order differential expression (1.6), its matrix formulation, and its quasi-derivative form will be made. The following theorem is easily established using Definition 1.4 and elementary methods.

THEOREM 1.2: Let  $Ly$  be given as in (1.6) and let  $A$  be given as in (1.5). Then for  $\lambda$  a complex number, the following are equivalent problems.

- (i)  $Ly = \lambda y$ ,
- (ii)  $y^{[2n]} = \lambda y$ ,
- (iii)  $y' = Ay$  where  $y = (y, y^{[1]}, \dots, y^{[2n-1]})^T$ .

A notational device that will prove useful will be introduced. This notation will be used to describe boundary conditions and to place complicated expressions into a very compact form.

DEFINITION 1.5: Let  $Ly$  be as given in (1.6). Let  $u$  and  $v$  be two



functions defined on the interval  $a < x < b$ , having quasi-derivatives with respect to  $L_y$  up to order  $2n - 1$ , and such that  $u$ ,  $v$ ,  $Lu$ , and  $Lv$  are in  $L^2(a, b)$ . Then define

$$[u v](x) = \sum_{j=0}^{n-1} (u^{[j]} \bar{v}^{[2n-1-j]} - u^{[2n-1-j]} \bar{v}^{[j]}). \quad (1.10)$$

Let  $L_y$  be as in (1.6). Then in the case  $n = 1$ ,  $L_y$  will be denoted by

$$L_y = - (py')' + qy$$

and in the case  $n = 2$ ,  $L_y$  will be denoted by

$$L_y = (ry'')'' - (py')' + qy.$$

The next theorem indicates some of the useful properties of the form (1.10). The proofs of these results follow from Definition 1.5 and by using integration by parts.

**THEOREM 1.3:** Let  $u$  and  $v$  be two complex-valued functions for which Definition 1.5 applies. Then

- (i)  $[u \bar{u}](x) = 0$ ,
- (ii)  $[u v](x) = \overline{[\bar{u} v]}(x)$ ,
- (iii)  $[u v](x) = - [\bar{v} \bar{u}](x)$ ,
- (iv)  $[\alpha u v](x) = \alpha [u v](x) = [u \bar{\alpha} v](x)$
- (v)  $[(u + v) w] = [u w] + [v w]$ , and
- (vi)  $\int_a^b (Lu) \bar{v} - u (L\bar{v}) dx = [u v](b) - [u v](a)$ .

Furthermore, if  $u$  and  $v$  are solutions of  $L_y = \lambda y$ , then

- (vii)  $[u \bar{v}](x) = \text{constant}$ , and
- (viii)  $2i \operatorname{Im} \lambda \int_a^b u(x) \bar{v}(x) dx = [u v](b) - [u v](a)$ .

In order to simplify the form of the Sturmian boundary conditions (1.7), the form (1.10) may be used. For simplicity, this will be done for the fourth-order case. The problems of other orders are similar. Consider one of the boundary conditions  $U_j y = 0$  and for the moment, write this in terms of quasi-derivatives as

$$\alpha y(a) + \beta y^{[1]}(a) + \gamma y^{[2]}(a) + \delta y^{[3]}(a) = 0. \quad (1.11)$$

It is desired to put the boundary condition (1.11) into the form

$$[y \phi](a) = 0 \quad (1.12)$$

where  $\phi(x)$  is, as yet, some unspecified function with a suitable number of quasi-derivatives. By using Definition 1.5, (1.12) may be written as

$$\phi^{[3]}(a)y(a) + \phi^{[2]}(a)y^{[1]}(a) - \phi^{[1]}(a)y^{[2]}(a) - \phi(a)y^{[3]}(a) = 0. \quad (1.13)$$

Comparing coefficients in (1.11) and (1.13), it is seen that  $\phi(x)$  need only be a solution of the differential equation  $Ly = \lambda y$  that satisfies the initial conditions at  $x = a$ ;

$$\phi(a) = -\delta, \quad \phi^{[1]}(a) = -\gamma, \quad \phi^{[2]}(a) = \beta, \quad \phi^{[3]}(a) = \alpha. \quad (1.14)$$

Theorem 1.1 states that there exists a unique function  $\phi$  satisfying the differential equation and the initial conditions (1.14). Thus, every boundary condition of the Sturmian type can be recast in the form (1.12). A Green's function for the formally self-adjoint expression (1.6) will be needed.

**DEFINITION 1.6:** Let  $Ly$  be given by (1.6) and boundary conditions of the form (1.7). Then a Green's function for the problem  $Ly = 0$  with

conditions (1.7) is a function  $G(x, z)$  satisfying the conditions:

(i)  $G(x, z)$  is continuous and has continuous quasi-derivatives with respect to  $x$  up to  $(2n-2)$  order inclusive for all  $x$  and  $z$  in the interval  $[a, b]$ .

(ii) For any fixed  $z$  in  $(a, b)$ ,  $G(x, z)$  has continuous quasi-derivatives of orders  $(2n-1)$  and  $2n$  with respect to  $x$  in each of the intervals  $[a, z)$  and  $(z, b]$ . The  $(2n-1)$ -st quasi-derivative is discontinuous at  $x = z$  with jump one. That is, for each fixed  $z$ ,

$$G^{[2n-1]}(z + 0, z) - G^{[2n-1]}(z - 0, z) = 1.$$

(iii) In each of the intervals  $[a, z)$  and  $(z, b]$ ,  $G(x, z)$  satisfies the differential equation  $Ly = 0$  and boundary conditions of the form (1.7).

**THEOREM 1.4:** If the boundary value problem  $Ly = 0$  with boundary conditions (1.7) has only the trivial solution, then the problem  $Ly = 0$  with conditions (1.7) has a unique Green's function.

A discussion of this last definition and theorem may be found in the book by Coddington and Levinson [7].

It is noted here how the word singular as applied to a differential expression will be used. An endpoint of an interval  $(a, b)$  will be called a singular endpoint of that interval with respect to the differential expression  $Ly$  if that endpoint is infinite or if at least one of the coefficients of  $Ly$  does not have a finite limit at that endpoint. Otherwise, the endpoint is called regular. A differential expression is called singular if at least one of the endpoints of its interval of definition is singular. It is called regular, otherwise. Notationally,

if it is intended that an endpoint of an interval be regular, it will be indicated as a closed endpoint. For example, with respect to  $Ly$ , the interval  $[a, b)$  has the regular endpoint  $a$  and the singular endpoint  $b$ .

In the following definition and theorem, the term eigenvalue will be defined and an existence theorem will be given. Various forms of arguments for the theorem may be found in standard books on ordinary differential equations such as Coddington and Levinson [7]. An easily followed and elegant development of the existence of eigenvalues for this problem is in the form of some unpublished notes by Lazer [62]. He has used the theory of the completely continuous symmetric operators. These notes deal only with the second order case, but the generalization to the  $2n$ -th order case is simple.

**DEFINITION 1.7:** Let  $Ly$  be the formally self-adjoint differential expression (1.6) with the coefficients  $p_k$  continuous and real-valued on the compact interval  $a \leq x \leq b$  with  $p_0(x)$  positive for all those  $x$ . Then, if  $\lambda$  is such that there exists a nontrivial solution for the self-adjoint problem (1.9), then  $\lambda$  is called an eigenvalue for (1.9) and the nontrivial solutions for that  $\lambda$  are called the eigenfunctions associated with  $\lambda$ .

**THEOREM 1.5:** Consider the self-adjoint problem described in the previous definition. Then the set of eigenvalues for this problem forms an infinite sequence of real numbers  $\lambda_k$ ,  $k = 1, 2, \dots$ . Furthermore, the eigenvalues may be ordered so that

$$\lambda_1 \leq \lambda_2 \leq \dots, \lim_{k \rightarrow \infty} |\lambda_k| = +\infty$$

This completes the results and definitions necessary in order to consider the limit-point and limit-circle cases of formally self-adjoint differential expressions.

## CHAPTER II

### THE SECOND-ORDER CASE

Consider the second-order formally self-adjoint differential equation

$$Ly = - (py')' + qy = \lambda y \quad (2.1)$$

where  $p$  and  $q$  are continuous and real-valued on the interval  $[0, \infty)$  and  $p$  takes on only positive values. Separate consideration of this case has the advantage of simplicity which is lacking in the fourth or  $2n$ -th order case,  $n > 1$ . The interval  $[0, \infty)$  will be considered only for convenience. By a suitable transformation of the real line and obvious modifications of the expressions, the following theorem will apply also to any half-open interval  $[a, b)$  or  $(a, b]$  where the open endpoint is singular and the closed endpoint is regular. In the following theorem, the functions  $\phi$  and  $\chi$  will be the boundary value functions as discussed in Chapter I. Also note here that the second-order quasi-derivatives with respect to the expression  $Ly$  in (2.1) have the form

$$y^{[1]} = py', \quad y^{[2]} = qy - (py')'.$$

The procedure will be to consider the problem (2.1) on a compact interval  $[0, b]$  where both endpoints are regular and then to move to the singular case by letting  $b$  tend to infinity.

THEOREM 2.1: Let  $b$  be a positive number and  $\lambda = u + iv$ ,  $v \neq 0$ , a complex number. Let  $\phi(x, \lambda, b)$  and  $\chi(x, \lambda, b)$  be two nontrivial solutions of (2.1) on the interval  $0 \leq x < \infty$  such that  $\phi$  takes real-valued constant initial conditions at  $x = 0$  and similarly for  $\chi$  except that the initial values are taken at  $x = b$ . Let  $\theta(x, \lambda, b)$  be a solution of (2.1) on that interval such that

$$[\phi \ \bar{\theta}] = 1 \quad (2.2)$$

and such that  $\theta$  takes real initial conditions at  $x = 0$ . Then the set of complex numbers  $\ell(b, \lambda)$  for which the solution

$$\psi(x, \lambda, b) = \theta(x, \lambda, b) + \ell(b, \lambda)\phi(x, \lambda, b) \quad (2.3)$$

satisfies the boundary conditions

$$[\psi \ \chi](b) = 0 \quad (2.4)$$

forms a circle  $C(b, \lambda)$  in the complex plane. Furthermore, as  $b$  tends to infinity (possibly through a sequence) the solution  $\psi(x, \lambda, b)$  tends to a function  $\psi(x, \lambda)$  which is a solution of (2.1) on the half-line  $0 \leq x < \infty$  that is also in  $L^2(0, \infty)$ . Also, the circles  $C(b, \lambda)$  tend to a set which is either a circle or a point, denoted by  $C(\lambda)$  and  $m(\lambda)$ , respectively.

Proofs of the preceding result are found in Chapter 9 of [7] and Chapter 2 of [74]. These proofs involve Möbius transformations and the arguments are easily followed. In the case the limiting set is the point  $m(\lambda)$ , the method of the proof established that  $\phi(x, \lambda) = \phi(x, \lambda, b)$  is not in  $L^2(0, \infty)$ , implying that not all solutions of  $Ly = \lambda y$  are in  $L^2(0, \infty)$ . If the limiting set is the circle  $C(\lambda)$ , then  $\phi(x, \lambda)$  is also

in  $L^2(0, \infty)$ , and thus all solutions of  $Ly = \lambda y$  are in  $L^2(0, \infty)$ . The existence of at least one solution of (2.1) in  $L^2(0, \infty)$  is dependent upon  $\lambda$  having a nonzero real part. This is illustrated by an example.

EXAMPLE 2.1: Let  $Ly = -y''$  and  $\lambda = 0$ . Then  $y_1(x) = 1$  and  $y_2(x) = x$  are linearly independent solutions of  $Ly = \lambda y$  and clearly no nontrivial linear combination of  $y_1$  and  $y_2$  can be in  $L^2(0, \infty)$ . As will be seen for this example, if  $\text{Im } \lambda \neq 0$ , then exactly one (up to linear independence) solution can be found for  $Ly = \lambda y$  that lies in  $L^2(0, \infty)$ . The existence of at least one  $L^2(0, \infty)$  solution for  $\text{Im } \lambda \neq 0$  is given by Theorem 2.1.

In the case the limit set is  $m(\lambda)$  the limit-point case is said to hold and in the case that the limit set is  $C(\lambda)$  the limit-circle case is said to hold. The next theorem will justify the statement at the end of Example 2.1 by showing that the terms limit-point and limit-circle are independent of the choice of  $\lambda$ , so long as  $\text{Im } \lambda \neq 0$ . That is, the presence of the limit-point or limit-circle case depends only upon the coefficients  $p$  and  $q$ .

THEOREM 2.2: (First Weyl Theorem) Let  $\lambda_0$  be any complex number and suppose that all solutions of

$$Ly = \lambda_0 y \tag{2.5}$$

are in  $L^2(0, \infty)$ . Then for any complex  $\lambda$ , all solutions of (2.1) are in  $L^2(0, \infty)$ .

PROOF: Let  $\phi(x)$  and  $\psi(x)$  be two linearly independent solutions of (2.5) such that  $[\phi \overline{\psi}] = 1$ . Then by the variation of parameters (for example, Coddington and Levinson [7:p. 87] or Naimark [65:p. 59])



the general solution of

$$(L - \lambda_0)y = (\lambda - \lambda_0)y \quad (2.6)$$

is given by the function

$$\begin{aligned} \chi(x, \lambda) = c_1 \phi(x) + c_2 \psi(x) + \\ \int_0^x [\phi(x)\psi(s) - \phi(s)\psi(x)](\lambda - \lambda_0)\chi(s, \lambda) ds, \end{aligned} \quad (2.7)$$

where it is assumed  $[\phi \overline{\psi}] = 1$ . For convenience, denote

$$(\int_c^x |f(s)|^2 ds)^{1/2} = I(f, c) \quad (2.8)$$

for  $x \geq c \geq 0$  and functions  $f$  for which the expression makes sense.

Since  $\phi$  and  $\psi$  are in  $L^2(0, \infty)$ , choose  $M > 0$  such that for all  $x \geq c$

$$I(\phi, c) < M, \quad I(\psi, c) < M. \quad (2.9)$$

An application of Schwarz' inequality, while using (2.8) and (2.9), implies

$$\begin{aligned} \left| \int_c^x [\phi(x)\psi(s) - \phi(s)\psi(x)](\lambda - \lambda_0)\chi(s, \lambda) ds \right| \\ \leq M(|\phi(x)| + |\psi(x)|)|\lambda - \lambda_0|I(\chi, c). \end{aligned} \quad (2.10)$$

By applying Minkowski's inequality to (2.7) and using (2.10), it follows that

$$I(/, c) \leq (|c_1| + |c_2|)M + |\lambda - \lambda_0|2M^2I(\chi, c) \quad (2.11)$$

Since  $I(\phi, c)$  and  $I(\chi, c)$  tend to zero as  $c$  tends to infinity, let  $c$  be sufficiently large so that  $M$  may be chosen to satisfy the condition

$|\lambda - \lambda_0|M^2 < 4^{-1}$ . Then for  $c$  sufficiently large, (2.11) may be written

$$I(\chi, c) \leq (|c_1| + |c_2|)^M + (1/2)I(\chi, c),$$

that is,

$$I(\chi, c) \leq 2(|c_1| + |c_2|)^M. \quad (2.12)$$

But the right hand side of (2.12) is bounded as  $x \rightarrow \infty$  and this implies  $\chi$  is in  $L^2(0, \infty)$ . Thus, all solutions of (2.6) are in  $L^2(0, \infty)$ , and the proof of the theorem is complete.

By the comments in the proof of Theorem 2.2, the limit-circle case holds for  $\lambda$  arbitrary if it holds for any one particular  $\lambda$ . Thus, the limit-circle case is independent of  $\lambda$ . Note that  $\lambda$  is not restricted to being non-real in this theorem, so if the limit-circle case holds for a real  $\lambda$ , then the limit-circle case holds for all  $\lambda$ . Thus, in order to check whether the limit-point or the limit-circle case holds for the expression  $Ly$ , one might as well consider the solutions of  $Ly = 0$ . If all solutions of this differential equation are square-integrable, then the limit-circle case holds. Otherwise, the limit-point case holds. Thus, the equation in Example 2.1 is in the limit-point case by the above remarks. It should also be noted that the requirement that  $q$  be continuous is not used and it is sufficient to assume  $q$  satisfies some integrability condition such as being locally integrable on  $0 \leq x < \infty$ .

### CHAPTER III

#### THE FOURTH-ORDER CASE

An extension of the analysis of the preceding chapter will now be made to the fourth-order case where  $Ly$  is expressed as

$$Ly = (ry'')'' - (py')' + qy. \quad (3.1)$$

The coefficients  $r$ ,  $p$  and  $q$  are to be real-valued and continuous on the half-line  $0 \leq x < \infty$  with  $r(x) > 0$  for  $x \geq 0$ . The comments following (2.1) concerning singular endpoints also apply here. Some of the analysis for the fourth-order case is the same as in the second-order case, while some of it is more difficult. The difficulties arise from the higher dimension spaces involved and from the existence of cases "between" the limit-circle and limit-point cases as compared to the second-order case in which each expression (2.1) is either limit-point or limit-circle. The possibility of cases between limit-point and limit-circle in the cases of order higher than two were not considered by Sin and Windau and this caused the errors in their analyses. The argument is restricted to the fourth-order case, but the notation adopted will be such that the extension to  $2n$ -th order cases will be simple in concept, if not in detail. The notation will also be such that the points of contact with the second-order case can easily be seen. The argument will proceed in the same manner as the second-order case after some preliminary results are presented. The proof of the

first lemma is identical to the proof of the classical theorem on linear independence of solutions when quasi-derivatives are read as ordinary derivatives. A proof for ordinary derivatives may be found on page 83 of the book by Coddington and Levinson [7].

LEMMA 3.1: Let  $\phi_r$ ,  $1 \leq r \leq 4$  be four solutions to  $Ly = \lambda y$ ,  $L$  given in (3.1), and let  $W(\phi_1, \phi_2, \phi_3, \phi_4)$  denote the determinant of the matrix whose  $ij$ -th entry is  $\phi_j^{[i-1]}(x)$ . Then  $W(\phi_1, \phi_2, \phi_3, \phi_4) = 0$  for some  $x_0$  in the interval of definition of (3.1) if and only if the set of functions  $\{\phi_r\}$  is linearly dependent on that interval.

The determinant  $W$  in Lemma 3.1 is called the (generalized) Wronskian of the solutions  $\phi_r$  with respect to (3.1) and has properties similar to those of the usual Wronskian. The next lemma makes a connection between the Wronskian of a set of solutions and the bilinear form in Definition 1.5.

LEMMA 3.2: Let  $\phi_1, \phi_2, \phi_3, \phi_4$  be four solutions of  $Ly = \lambda y$ . Then

$$\begin{aligned} W(\phi_1, \phi_2, \phi_3, \phi_4) &= [\phi_1 \bar{\phi}_2] [\phi_3 \bar{\phi}_4] - [\phi_1 \bar{\phi}_3] [\phi_2 \bar{\phi}_4] \\ &\quad + [\phi_1 \bar{\phi}_4] [\phi_2 \bar{\phi}_3]. \end{aligned} \quad (3.2)$$

PROOF: By Definition 1.5,

$$[\phi_r \bar{\phi}_s] = \phi_r \phi_s^{[3]} + \phi_r^{[1]} \phi_s^{[2]} - \phi_r^{[2]} \phi_s^{[1]} - \phi_r^{[3]} \phi_s. \quad (3.3)$$

The lemma is established by substituting (3.3) into the right side of (3.2) and comparing this with the expansion of the determinant of the matrix  $W$ . The proof is then complete.

For functions  $f_1, f_2, \dots, f_n$  integrable on  $0 \leq x \leq b$ , let  $f$  denote the column matrix

$$f = (f_1, \dots, f_n)^T. \quad (3.4)$$

Let  $G(f; b)$  denote that  $n \times n$  matrix whose  $ij$ -th entry is the integral

$$\int_0^b f_i(x) \bar{f}_j(x) dx. \quad (3.5)$$

The matrix  $G(f; b)$  is called a Gram matrix. The following lemma concerning Gram matrices and determinants will be necessary in the next series of lemmas. This lemma is established as Lemma 1 of [25].

LEMMA 3.3: Let  $f$  be as in (3.4) and let  $V$  be an  $n \times n$  matrix of constants. Then

- (i)  $G(Vf; b) = VG(f; b)V^*$ ,
- (ii)  $\det G(Vf; b) = |\det V|^2 \det G(f; b)$ .

The following theorem is known as the Courant (or Poincaré) Minimax Principle. The proof is straightforward and may be found in Halmos [47] or Courant and Hilbert [9].

THEOREM 3.1: Let  $A$  be the matrix of a hermitian transformation on an  $n$ -dimensional complex inner product space  $V$ . Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  be the ordered eigenvalues of  $A$ . For each subspace  $U$  of  $V$  and  $1 \leq k \leq n$ , define

$$u(U) = \sup \{ z^* A z \mid z \text{ is in } U \text{ and } \sum_{j=1}^n |z_j|^2 = 1 \},$$

where  $z$  is a complex  $n$ -vector,  $z = (z_1, z_2, \dots, z_n)^T$  and define for each  $k$

$$u_k = \inf \{u(U) \mid \dim U = n - k + 1\}.$$

Then  $\lambda_k = u_k$  for each  $k$ ,  $1 \leq k \leq n$ .

The next lemma establishes a relationship between corresponding eigenvalues of two positive definite hermitian matrices. It is well known that the eigenvalues of a positive definite hermitian matrix are real and positive. This lemma will be used to establish a result that leads to the determination of the number of solutions of  $Ly = \lambda y$  that are in the class  $L^2(0, \infty)$ . The proof of this lemma follows by applying Theorem 3.1 to the matrices  $A$  and  $A + B$ .

**LEMMA 3.4:** Let  $A$  and  $B$  be positive definite hermitian matrices of size  $n$ . Suppose  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$  are the eigenvalues of  $A$  and  $\lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_n > 0$  are the eigenvalues of  $A + B$ . Then  $\lambda'_k \geq \lambda_k$  for  $k = 1, 2, \dots, n$ .

The following lemma is used in the theorem establishing the limit- $p$ ,  $2 \leq p \leq 4$  cases in the fourth order problem. This result states that, under certain conditions on the functions  $\phi_1$  and  $\phi_2$  and  $G(\phi; b)$  the Gram matrix of (3.5), the eigenvalues of  $G(\phi; b)$  are increasing functions of  $b$ .

**LEMMA 3.5:** Let  $\phi_1(x)$  and  $\phi_2(x)$  be any two complex-valued functions defined on the interval  $0 \leq x < \infty$  such that

- (i)  $\phi_1$  and  $\phi_2$  are linearly independent on the interval  $0 \leq x \leq b$  for all  $b > 0$ , and
- (ii)  $\phi_1$  and  $\phi_2$  are in  $L^2(0, \infty)$  locally.

For each  $b > 0$ , let  $G(\phi; b)$  be the matrix given by (3.5) with  $\phi = (\phi_1, \phi_2)^T$ . Let  $\lambda_1(b) \leq \lambda_2(b)$  denote the ordered eigenvalues of

$G(\emptyset; b)$ . Then the following statements are true.

(iii)  $G(\emptyset; b)$  is positive definite for each  $b > 0$ , and

(iv) for  $b' > b > 0$  and  $s = 1$  or  $2$ ,  $\lambda_s(b') \geq \lambda_s(b)$ .

PROOF: Let  $z = (z_1, z_2)^T$  be a nonzero complex 2-vector. Then

$$z^* G(\emptyset; b) z = \int_0^b |z_1 \phi_1(x) + z_2 \phi_2(x)|^2 dx.$$

Since  $z_1$  and  $z_2$  are not both zero and by the linear independence of the system  $\phi$ , it follows that  $z^* G(\emptyset; b) z > 0$ . Therefore,  $G(\emptyset; b)$  is positive definite and thus conclusion (iii) holds.

Statement (iv) is established in Section 8 of [26], completing the proof of the lemma.

The following lemma will be used in establishing the number of  $L^2(0, \infty)$  solutions of  $Ly = \lambda y$ . A proof of this lemma may also be found in [26].

LEMMA 3.6: Let  $\phi_1$  and  $\phi_2$  satisfy conditions (i) and (ii) of Lemma 3.5. Then for  $r = 1$  or  $r = 2$ , as functions of  $b$ , the quotients

$$C_r(b) = \frac{\int_0^b |\phi_{3-r}(x)|^2 dx}{\det G(\emptyset; b)}$$

are monotone decreasing as  $b$  is increasing.

The next lemma establishes another property of the functions  $C_r(b)$  of Lemma 3.6. In Lemma 3.6 it was shown that these functions are decreasing functions of  $b$  and the next lemma shows that they tend to strictly positive limits as  $b$  tends to infinity provided there are certain linear combinations of  $\phi_1$  and  $\phi_2$  in  $L^2(0, \infty)$ . A proof of this

result may be found in [26].

**LEMMA 3.7:** Suppose  $\phi_1$  and  $\phi_2$  satisfy conditions (i) and (ii) of Lemma 3.5. Then for  $r = 1$  or  $r = 2$ , the function  $C_r(b)$  as given in (3.6) tends to a strictly positive limit as  $b$  tends to infinity if and only if there exists a linear form  $\alpha_1\phi_1 + \alpha_2\phi_2$ , with  $\alpha_r$  not zero, which belongs to the class  $L^2(0, \infty)$ .

The following lemma connects the behavior of the eigenvalues of  $G(\phi; b)$  as  $b \rightarrow \infty$  with the number of linearly independent linear forms  $\alpha_1\phi_1 + \alpha_2\phi_2$  in  $L^2(0, \infty)$ . This lemma is Theorem 5 of [26].

**LEMMA 3.8:** Let  $\phi_1(x)$  and  $\phi_2(x)$  satisfy conditions (i) and (ii) of Lemma 3.5. For each  $b > 0$ , let  $\lambda_r$ ,  $r = 1, 2$  be the eigenvalues of the matrices  $G(\phi; b)$  with  $0 < \lambda_1(b) \leq \lambda_2(b)$ . Let  $S$  be the number of those  $\lambda_r(b)$  which tend to finite limits as  $b \rightarrow \infty$ . Then the number (up to linear independence) of linear forms  $\alpha_1\phi_1 + \alpha_2\phi_2$  in  $L^2(0, \infty)$  is  $S$ .

It is noted here that all the preceding lemmas that were stated for two functions  $\phi_1$  and  $\phi_2$  may be generalized to sets of  $n$  functions. These lemmas were stated for two functions since this is sufficient in the case under consideration. The generalization of the lemmas to sets of  $n$  functions may be found in the references given. The next theorem is useful for establishing some necessary identities in this chapter and in Chapters IV and V.

**THEOREM 3.2:** Let  $\{f_i\}$ ,  $\{g_i\}$ ,  $1 \leq i \leq 2n + 1$ , be any two sets of  $2n + 1$  functions, each having quasi-derivatives up through order  $2n - 1$  with respect to (1.6). Then



$$\det_{1 \leq i, j \leq 2n+1} ([f_i \ g_j]) = 0 \quad (3.7)$$

for all  $x$ , where the bilinear forms  $[f_i \ g_j]$  are with respect to the differential expression (1.6).

PROOF: For notational purposes, define

$$F^{[j]} = (f_1^{[j]}, f_2^{[j]}, \dots, f_{2n+1}^{[j]})^T. \quad (3.8)$$

Then the left side of (3.7) may be expressed as

$$\det(u_1, u_2, \dots, u_{2n+1}) \quad (3.9)$$

where by (1.10) and (3.8), each  $u_k$ ,  $1 \leq k \leq 2n+1$ , is the sum of  $2n$  terms,

$$u_k = \sum_{i=0}^{n-1} (F^{[i]} \underset{e_k}{-}^{[2n-1]} - F^{[2n-1]} \underset{e_k}{-}^{[i]}).$$

Using the bilinearity of the determinant functions with respect to columns, (3.9) is the sum of  $(2n)^{2n+1}$  terms, each of the form

$$\pm \det(v_1, v_2, \dots, v_{2n+1}) \quad (3.10)$$

where

$$v_k = F^{[N_k]} \underset{e_k}{-}^{[2n-1-N_k]}, \quad 1 \leq k \leq 2n+1, \quad (3.11)$$

and the  $N_k$  are chosen from the set of integers  $\{0, 1, \dots, 2n-1\}$ .

The expression (3.10) may be expressed by using (3.11) as

$$\pm \prod_{k=1}^{2n+1} \underset{e_k}{-}^{[2n-1-N_k]} \det(F^{[N_1]}, \dots, F^{[N_{2n+1}]}) \quad (3.12)$$

However, from the above description of the  $N_k$ , there are  $2n+1$

integers chosen from a set of  $2n$  integers. Thus, there must be at

least one pair of the  $N_k$  that are identical, say  $N_j = N_k$ . Therefore, at least two of the columns of the determinant in (3.12) must be identical. By the well-known properties of determinants, (3.12) is zero. The conclusion (3.7) then follows and the proof of the lemma is complete.

The next lemma is an application of Theorem 3.2 and establishes the value of a determinant that is necessary later.

LEMMA 3.9: Let  $\phi_1, \phi_2, \theta_1$ , and  $\theta_2$  be four functions that are defined on  $0 \leq x < \infty$  with three finite quasi-derivatives with respect to (3.1) on that interval. Suppose further that these functions satisfy the differential equation

$$Ly = \lambda y, \text{ Im } \lambda \neq 0$$

with  $Ly$  given by (3.1) and also satisfy the conditions

$$[\theta_1 \bar{\theta}_2] = 0, [\phi_r \bar{\theta}_s] = \delta_{rs}, [\phi_1 \bar{\phi}_2] = 0.$$

For each  $b > 0$  and  $r = 1$  or  $2$ , let  $B_r(b)$  be the matrix

$$B_r(b) = \begin{bmatrix} [\phi_1 \phi_1] & [\phi_1 \phi_2] & [\phi_1 \theta_r] \\ [\phi_2 \phi_1] & [\phi_2 \phi_2] & [\phi_2 \theta_r] \\ [\theta_r \phi_1] & [\theta_r \phi_2] & [\theta_r \theta_r] \end{bmatrix}$$

with all the entries in the matrix evaluated at  $x = b$ . Then

$$\det B_r(b) = [\phi_{3-r} \phi_{3-r}](b), \quad r = 1, 2.$$

The proof of the preceding lemma is established by making the substitutions  $f_1 = g_1 = \bar{f}_4 = \bar{g}_4 = \phi_1$ ,  $f_2 = g_2 = \bar{f}_5 = \bar{g}_5 = \phi_2$ , and  $f_3 = g_3 = \theta_r$  into Theorem 3.2 and using the hypotheses of the lemma. The following theorem establishes an interesting relationship between the Wronskian of the four boundary value functions for the fourth-order problem and the eigenvalues of the problem. This result is used to construct the solutions to the fourth-order problem that are in  $L^2(0, \infty)$ .

**THEOREM 3.3:** Let  $b$  be a positive real number and  $\lambda = u + iv$  a complex number. Let  $\phi_r(x, \lambda)$ ,  $\chi_r(x, \lambda, b)$ ,  $r = 1, 2$ , be four solutions of

$$Ly = \lambda y \quad (3.13)$$

on  $0 \leq x < \infty$  with  $L$  given in (3.1) such that the  $\phi_r$  take constant real-valued and independent initial conditions at  $x = 0$  and similarly for the  $\chi_r$  except that the initial conditions are taken at  $x = b$ . In all cases, the initial conditions are to be independent of  $\lambda$ . Suppose further that the following conditions hold;

$$[\phi_1 \bar{\phi}_2] = 0, [\chi_1 \bar{\chi}_2] = 0. \quad (3.14)$$

Let  $D$  be defined by

$$D(\lambda, b) = [\phi_1 \bar{\chi}_1] [\phi_2 \bar{\chi}_2] - [\phi_1 \bar{\chi}_2] [\phi_2 \bar{\chi}_1] \quad (3.15)$$

where the explicit dependence of the right hand side of (3.15) upon  $b$  and  $\lambda$  is suppressed. Then, as a function of  $\lambda$ ,  $D(\lambda, b)$  is entire, its zeros are real and those zeros are the eigenvalues of the problem defined by (3.13) and (3.14) with the boundary conditions on a solution  $y$

$$[\phi_r y](0) = 0, [\chi_r y](b) = 0, r = 1, 2 \quad (3.16)$$

PROOF: By Lemma 3.2

$$W(\phi_1, \phi_2, \chi_1, \chi_2) = [\phi_1 \bar{\phi}_2] [\chi_1 \bar{\chi}_2] - [\phi_1 \bar{\chi}_1] [\phi_2 \bar{\chi}_2] \\ + [\phi_1 \bar{\chi}_2] [\phi_2 \bar{\chi}_1].$$

Applying (3.14), it then follows that

$$W(\phi_1, \phi_2, \chi_1, \chi_2) = -D \quad (3.17)$$

Suppose that for some value of  $\lambda$ , say  $\lambda = \lambda_0$ ,

$$-D(\lambda_0) \neq 0. \quad (3.18)$$

Then by Lemma 3.1 and (3.17), the four functions  $\phi_r, \chi_r, r = 1, 2$ , form a fundamental set of solutions of the eigenvalue problem (3.13), with  $\lambda$  replaced by  $\lambda_0$ . Thus, if  $y$  is a solution to this problem, then

$$y(x) = \sum_{i=1}^2 (\alpha_i \phi_i(x) + \beta_i \chi_i(x)). \quad (3.19)$$

Letting  $r = 1$  and 2 and application of the four boundary conditions (3.16) to the solution (3.19) yields the following system of four equations in the four unknowns  $\alpha_i, \beta_i, i = 1, 2$ :

$$\sum_{i=1}^2 (\alpha_i [\phi_r \bar{\phi}_i] + \beta_i [\phi_r \bar{\chi}_i]) = 0, \\ \sum_{i=1}^2 (\alpha_i [\chi_r \bar{\phi}_i] + \beta_i [\chi_r \bar{\chi}_i]) = 0 \quad (3.20)$$

where  $r = 1$  and 2. By the use of Theorem 1.4 and the conditions (3.14), the determinant of the coefficient matrix of the system (3.20) is seen to be  $-D(\lambda_0)$ . But from (3.18),  $-D(\lambda_0) \neq 0$ . Thus the system (3.20) has no nontrivial solution for the  $\alpha_i$  and  $\beta_i$ , and  $y(x) = 0$ . That is,  $\lambda_0$  is not an eigenvalue for the described problem.

Suppose for some value of  $\lambda$ , say  $\lambda = \lambda_0$ ,  $D(\lambda_0) = 0$ . Then by (3.17) and Lemma 3.1, it follows that the four functions  $\phi_1$ ,  $\phi_2$ ,  $\chi_1$ , and  $\chi_2$  are linearly dependent. Thus there is a linear combination of these functions that is zero. That is, there exist  $\alpha_i$ ,  $\beta_i$ ,  $i = 1, 2$ , with not both of the  $\alpha_i$  zero and not both of the  $\beta_i$  zero, such that

$$\sum_{i=1}^2 (\alpha_i \phi_i(x) - \beta_i \chi_i(x)) = 0. \quad (3.21)$$

Let  $y$  be defined by

$$y(x) = \alpha_1 \phi_1(x) + \alpha_2 \phi_2(x).$$

Then, clearly,  $y$  satisfies (3.13). Also by Theorem 1.3 and (3.14)

$$[\phi_r y](0) = \alpha_1 [\phi_r \bar{\phi}_1](0) + \alpha_2 [\phi_r \bar{\phi}_2](0) = 0.$$

By the definition of  $y$  and (3.21)

$$y(x) = \beta_1 \chi_1(x) + \beta_2 \chi_2(x)$$

and by the second part of (3.14) and Theorem 1.3

$$[\chi_r y](0) = \beta_1 [\chi_r \bar{\chi}_1](0) + \beta_2 [\chi_r \bar{\chi}_2](0) = 0.$$

Therefore,  $y$  is a nontrivial solution to the eigenvalue problem as described and thus  $\lambda_0$  is an eigenvalue of this problem. The conclusion that the zeros of  $D$  are the eigenvalues of the problem (3.13), (3.14), and (3.15) then follows.

By Theorem 1.1, the functions  $D(\lambda, b)$  as functions of  $\lambda$  are entire and by Theorem 1.5, the eigenvalues are all real. Therefore, the zeros of  $D(\lambda, b)$  are all real. This completes the proof of Theorem 3.3.

This completes the preliminary results necessary to state and establish a series of theorems that are the analogies of the theorems in the second-order case. In the following series of theorems, the functions  $\phi_1, \phi_2, \theta_1, \theta_2, \chi_1$ , and  $\chi_2$  will play the same roles as did the functions  $\phi, \theta$ , and  $\chi$  in the second-order case. Also, there will arise a set of coefficients  $l_{rs}$ ,  $r, s = 1, 2$ , that are the analogies of the coefficient  $l$  in the second-order case. As expected from the large number of lemmas necessary for the fourth-order case, the analysis is more difficult, but similar. Those preceding lemmas that are specific to the fourth-order case have generalizations to differential expressions of any even order. In general, the arguments for these generalizations are no more difficult than those for the fourth-order case. The further restriction of these lemmas to the second-order case is also valid, but the second-order case was presented separately since the use of these lemmas obscures the elegance of the analysis. The proof of the following theorem is contained in Sections 16, 17, and 18 of [30].

**THEOREM 3.4:** Let  $b$  be a positive real number and  $\lambda$  a complex number,  $\lambda = u + iv$ , with  $v \neq 0$ . Let  $\phi_r(x, \lambda), \chi_r(x, \lambda, b)$ ,  $r = 1, 2$ , be four solutions of  $Ly = \lambda y$ , with  $L$  given in (3.1), on the interval  $0 \leq x < \infty$  such that  $\phi_1$  and  $\phi_2$  take real-valued independent initial conditions at  $x = 0$  and similarly for  $\chi_1$  and  $\chi_2$  except that the initial conditions are taken at  $x = b$ . In all cases, the initial conditions are to be independent of  $\lambda$ . Suppose  $\phi_1, \phi_2, \chi_1$ , and  $\chi_2$  satisfy the conditions (3.14).

Let  $\theta_r(x, \lambda)$ ,  $r = 1, 2$ , be two solutions of (3.13) on  $0 \leq x < \infty$  that take constant real-valued initial conditions at  $x = 0$  in such a

way that

$$[\theta_1 \bar{\theta}_2] = 0, [\phi_r \bar{\theta}_s] = \delta_{rs}$$

for  $r, s = 1, 2$ , where  $\delta_{rs}$  is the Kronecker delta. Then there exists two solutions  $\psi_r(x, \lambda, b)$ ,  $r = 1, 2$ , of  $Ly = \lambda y$  that satisfy the boundary conditions

$$[\psi_r \chi_s](b) = 0, 1 \leq r, s \leq 2$$

and are expressible in the form

$$\psi_r(x, \lambda, b) = \theta_r(x, \lambda) + \sum_{s=1}^2 l_{rs}(\lambda, b) \phi_s(x, \lambda) \quad (3.22)$$

for  $r = 1, 2$ . Furthermore, for each  $r$ , the set of pairs  $(l_{r1}, l_{r2})$  of complex numbers in (3.22), when considered as points in the two-dimensional complex space  $\mathbb{C}^2$ , all lie on a closed and bounded hypersurface, say  $S_r(\lambda, b)$ , as the functions  $\chi_r$  range over all allowable initial values at  $x = b$ . Also, the interior of  $S_r(\lambda, b)$ , together with its boundary, forms an ellipsoid, say  $E_r(b)$ , when considered as a subset of  $\mathbb{R}^4$  and  $E_r(b)$  is of dimension four in  $\mathbb{R}^4$ .

The next theorem states that, as  $b$  tends to infinity, the sets  $E_r(b)$  are nested and in fact, tend to a set  $E_r(\infty)$  which is of dimension zero, two, or four in  $\mathbb{R}^4$ . These dimensions will be the deciding factor as to the number of  $L^2(0, \infty)$  solutions of  $Ly = \lambda y$ . The analogy in the second-order case is the set of circles in the complex plane  $\mathbb{C}$  which tend to boundaries of sets of dimension zero or two in the Cartesian space  $\mathbb{R}^2$ . A proof of this result may be found in Section 17 of [30].

**THEOREM 3.5:** Let the conditions and hypotheses of Theorem 3.4 hold for each  $b > 0$  and for a fixed  $\lambda$ ,  $\text{Im } \lambda \neq 0$ . For  $r = 1$  and  $r = 2$ , let  $E_r(b)$  and  $S_r(b)$  be those sets described in Theorem 3.4. Then as  $b$  tends to infinity, each ellipsoid  $E_r(b)$  tends to a convex set, say  $E_r(\infty)$ , which is of dimension zero, two, or four in  $\mathbb{R}^4$ . Furthermore, for  $S$  as defined in Lemma 3.8 and  $C_r(b)$  defined in (3.6),  $\dim E_r(\infty)$  is zero if  $C_r(\infty)$  is zero and  $\dim E_r(\infty) = 2S$  if  $C_r(\infty)$  is positive, where  $C_r(\infty)$  denotes  $\lim_{b \rightarrow \infty} C_r(b)$ .

The following theorem is a consequence of the two preceding theorems. The analogy of this theorem in the second-order case is a part of Theorem 2.1. As in the second-order case, for  $\text{Im } \lambda \neq 0$ , the number of linearly independent  $L^2(0, \infty)$  solutions of  $Ly = \lambda y$  is at least half the order of the expression  $Ly$ . As commented before, the necessary theorems for the result above hold for any even order expression of the form (1.6) with real-valued continuous coefficients. A proof of this theorem is given in Section 18 of [30].

**THEOREM 3.6:** Let the conditions and hypotheses of Theorems 3.4 and 3.5 hold for each  $b > 0$ . Let  $\lambda = u + iv$  be a complex number with  $v \neq 0$ . Then there exists at least two (up to linear independence) solutions of the differential equation  $Ly = \lambda y$  that lie in  $L^2(0, \infty)$ ,  $Ly$  given in (3.1).

The following theorem relates the dimension of the sets  $E_r(\infty)$  to the number of  $L^2(0, \infty)$  solutions of  $Ly = \lambda y$ . This is done by relating the dimensions of the sets  $E_r(\infty)$ ,  $r = 1, 2$ , to the number of linearly independent forms on  $\phi_1$  and  $\phi_2$  that are in  $L^2(0, \infty)$ . This, together



with the knowledge that  $\varphi_r$  is in  $L^2(0, \infty)$  for  $r = 1$  and  $2$ , will determine the  $L^2$  character of the solution space.

**THEOREM 3.7:** Let the conditions, hypotheses, and notation of the preceding three theorems hold for a fixed  $\lambda$ ,  $\text{Im } \lambda \neq 0$ . For  $r = 1$  and  $r = 2$ , let  $D = (d_1, d_2)$  be that pair of positive integers defined by

$$d_r = (1/2)\dim E_r(\infty).$$

Let  $L^2 = L^2(0, \infty)$  and let  $K$  denote the maximum number (up to linear independence) of  $L^2$  solutions of  $Ly = \lambda y$ . Then the following statements hold.

- (i)  $D = (2, 2)$  if and only if  $\phi_1$  and  $\phi_2$  are both in  $L^2$ .
- (ii) The four cases  $D = (0, 2)$ ,  $(1, 2)$ ,  $(2, 1)$ , and  $(2, 0)$  cannot occur.
- (iii)  $D = (1, 0)$  if and only if  $\phi_1$  is  $L^2$  and  $\phi_2$  is not  $L^2$ .
- (iv)  $D = (0, 1)$  if and only if  $\phi_1$  is not  $L^2$  and  $\phi_2$  is  $L^2$ .
- (v)  $D = (1, 1)$  if and only if neither  $\phi_1$  nor  $\phi_2$  is in  $L^2$  but there is a linear form  $\alpha_1\phi_1 + \alpha_2\phi_2$  ( $\alpha_1$  and  $\alpha_2$  not zero) which is  $L^2$ .
- (vi)  $D = (0, 0)$  if and only if neither  $\phi_1$  nor  $\phi_2$  is  $L^2$  and no nontrivial linear form on  $\phi_1$  and  $\phi_2$  is  $L^2$ .

Furthermore, in (i),  $K = 4$ ; in (iii)-(v),  $K = 3$ ; in (vi),  $K = 2$ .

**PROOF:** It should be noted that by Theorem 3.5,  $d_r$  takes on only the values zero, one, or two. Thus, all possible values of  $D$  are included in (i)-(vi). Also, by Theorem 3.6,  $K \geq 2$ . Since  $K \leq 4$ , all possible values of  $K$  are also considered in the statement of the theorem. As in Theorem 3.5,  $S$  will denote the number of eigenvalues of  $G(\phi; b)$  having finite limits as  $b \rightarrow \infty$ .

In the proof of this theorem, it will be necessary to make frequent reference to several statements. For convenience, they are stated as:

- (a)  $d_r = 0$  if and only if  $C_r(\infty) = 0$ ,
- (b) if  $C_r(\infty) > 0$ , then  $d_r = S$ , and
- (c) if  $d_r > 0$ , then  $C_r(\infty) > 0$  and  $d_r = S$ .

Some of these statements have been established while others require some argument. In (a), if  $C_r(\infty) = 0$ , then  $d_r = 0$  by Theorem 3.5. For the converse, suppose  $d_r = 0$ . If  $C_r(\infty) > 0$ , then an application of Lemma 3.7 followed by Lemma 3.8 implies  $S > 0$ . Then one of the cases (ii) or (iii) of Theorem 3.5 holds and  $d_r = 1$  or  $2$ , a contradiction. Thus,  $C_r(\infty) = 0$  and (a) holds. Statement (b) follows from Theorem 3.5. Statement (c) follows from (a) and (b).

Each of the statement (i) through (vi) will be considered separately. Suppose  $D = (2,2)$ . By (c),  $S = d_r = 2$  and Lemma 3.8 implies there are two linearly independent linear forms of  $\phi_1$  and  $\phi_2$  that are in  $L^2$ . Thus, both  $\phi_1$  and  $\phi_2$  are in  $L^2$ . For the converse, suppose both  $\phi_1$  and  $\phi_2$  are in  $L^2$ . Lemma 3.8 implies  $S = 2$  and by Lemma 3.7, both  $C_r(\infty)$  are positive. Thus, (c) implies  $D = (2,2)$ . Therefore, (i) holds.

Let  $r$  be either 1 or 2 and suppose  $d_r = 2$ . By (c),  $S = 2$  and using the argument for statement (i), both  $\phi_1$  and  $\phi_2$  are in  $L^2$  and so  $D = (2,2)$ . That is,  $d_{3-r} = 2$ . Thus, the four possibilities  $D = (0,2)$ ,  $(1,2)$ ,  $(2,1)$ , and  $(2,0)$  cannot occur and (ii) follows.

Suppose  $D = (1,0)$ . Then  $S = 1$  by (c) and Lemma 3.8 implies there is exactly one nontrivial linear form  $\alpha_1\phi_1 + \alpha_2\phi_2$  that is in  $L^2$ . By (a) and Lemma 3.7,  $C_1(\infty) > 0$  implies  $\alpha_1 \neq 0$  and  $C_2(\infty) = 0$  implies

$\alpha_2 = 0$ . Thus,  $\phi_1$  is in  $L^2$  and  $\phi_2$  is not in  $L^2$ . ( $\phi_2$  cannot be in  $L^2$ , for (i) would then imply  $D = (2,2)$ .) Conversely, suppose  $\phi_1$  is  $L^2$  and  $\phi_2$  is not  $L^2$ . If  $\alpha_1\phi_1 + \alpha_2\phi_2$  is any nontrivial linear form in  $L^2$ , then  $\alpha_2 = 0$ . Thus, by Lemma 3.7,  $C_1(\infty) > 0$  and  $C_2(\infty) = 0$ , that is,  $D = (S,0)$  by (a) and (b). By Lemma 3.8,  $\phi_1$  in  $L^2$  implies  $S > 0$ . Therefore,  $D = (1,0)$  or  $D = (2,0)$ . The second possibility is impossible by (ii), and so  $D = (0,1)$  and (iii) holds. Statement (iv) is completely analogous.

Suppose  $D = (1,1)$ . Both  $C_r(\infty)$  are positive and  $S = 1$  by (c). Thus, there is exactly one nontrivial linear form  $\alpha_1\phi_1 + \alpha_2\phi_2$  in  $L^2$  and by Lemma 3.7, neither  $\alpha_1$  nor  $\alpha_2$  can be zero. For if either  $\phi_1$  or  $\phi_2$  is in  $L^2$ , then by the independence of  $\phi_1$  ( $\phi_2$ ) and  $\alpha_1\phi_1 + \alpha_2\phi_2$ ,  $\phi_2$  ( $\phi_1$ ) must also be in  $L^2$ , implying  $D = (2,2)$  by (i), a contradiction. Thus, neither  $\phi_1$  nor  $\phi_2$  is in  $L^2$ . Conversely, suppose neither  $\phi_1$  nor  $\phi_2$  is in  $L^2$ , but there is a nontrivial form  $\alpha_1\phi_1 + \alpha_2\phi_2$  that lies in  $L^2$ . Since neither  $\phi_1$  nor  $\phi_2$  are in  $L^2$ ,  $\alpha_1 \neq 0$  and  $\alpha_2 \neq 0$ . By Lemma 3.8,  $S = 1$  and by Lemma 3.7, both  $C_r(\infty)$  are positive. Therefore, applying (b),  $D = (1,1)$  and (v) must hold.

Finally, assume  $D = (0,0)$ . Then, both  $C_r(\infty)$  are zero by (a). Since  $C_1(\infty) = 0$ , Lemma 3.7 implies no linear form  $\alpha_1\phi_1 + \alpha_2\phi_2$  with  $\alpha_1 \neq 0$  can be in  $L^2$ . Thus,  $\phi_1$  is not in  $L^2$ . Similarly,  $\phi_2$  is not in  $L^2$ . Therefore, neither  $\phi_1$  nor  $\phi_2$  nor any nontrivial linear form  $\alpha_1\phi_1 + \alpha_2\phi_2$  can be in  $L^2$ . For the converse, suppose neither  $\phi_1$  nor  $\phi_2$  nor any nontrivial linear form  $\alpha_1\phi_1 + \alpha_2\phi_2$  is in  $L^2$ . By Lemma 3.7 and (a),  $D = (0,0)$  and (vi) holds.

By the remarks at the beginning of the proof,  $K$  must take one of the values two, three, or four. Clearly,  $K = 4$  if and only if  $\phi_1$  and

$\phi_2$  are both in  $L^2$ . Thus,  $K = 4$  in case (i). In each of the cases (iii)-(v), there is one solution not in  $L^2$  implying  $K < 4$  and there is one solution linearly independent of  $\psi_1$  and  $\psi_2$  in  $L^2$ , and so  $K > 2$ , implying  $K = 3$ . Conversely, if  $K = 3$ , then clearly not both  $\phi_1$  and  $\phi_2$  are in  $L^2$ . Suppose no nontrivial linear form  $\alpha_1\phi_1 + \alpha_2\phi_2$  is in  $L^2$ .  $K = 3$  and  $\psi_1, \psi_2, \phi_1$ , and  $\phi_2$  form a fundamental set of solutions, so there are three linearly independent solutions that lie in  $L^2$  of the form

$$\alpha_{k1}\psi_1 + \alpha_{k2}\psi_2 + \alpha_{k3}\phi_1 + \alpha_{k4}\phi_2, \quad 1 \leq k \leq 3.$$

Each of  $\psi_1$  and  $\psi_2$  is in  $L^2$ , so each of the three linear forms

$$\alpha_{k3}\phi_1 + \alpha_{k4}\phi_2, \quad 1 \leq k \leq 3 \tag{3.23}$$

must be in  $L^2$ . But, by the supposition that no nontrivial linear form  $\alpha_1\phi_1 + \alpha_2\phi_2$  is in  $L^2$ , each of these must be trivial. Thus the three forms

$$\alpha_{k1}\psi_1 + \alpha_{k2}\psi_2, \quad 1 \leq k \leq 3$$

are linearly independent, an obvious contradiction. Thus, there must be at least one of the forms (3.23) that is nontrivial, and so one of the cases (iii), (iv), or (v) must hold. Suppose case (vi) holds.

Then clearly,  $2 \leq K < 4$ . By the argument above, if it is assumed that  $K = 3$ , then one of the cases (iii)-(v) must hold, contrary to supposition. Thus,  $K = 2$ . Conversely, if  $K = 2$ , then clearly no nontrivial linear combination on  $\phi_1$  and  $\phi_2$  can lie in  $L^2$  for this would imply  $K > 2$ . Thus, case (vi) holds. This completes the proof of the theorem.

In Theorem 3.7, case (i) is called the limit-circle or limit-4 case and case (vi) is called the limit-point case. It is these two situations which are analogous to the limit-circle and limit-point cases, respectively, in the second-order problem. It is now desired to give the analogy of Theorem 2.2. That is, for  $\text{Im } \lambda \neq 0$ , the number of  $L^2(0, \infty)$  solutions of  $Ly = \lambda y$  is invariant. As to be expected, the fourth-order case is more complicated. The primary difficulty is the existence of cases "between" the limit-point and limit-circle cases, a nonexisting problem in the second-order case. Theorem 2.2 used the method of variation of parameters to show that if all solutions of  $Ly = \lambda y$  are in  $L^2(0, \infty)$  for a particular  $\lambda$ , then the same holds for all  $\lambda$ . This theorem was sufficient in the second-order case to establish the invariance of the number of  $L^2(0, \infty)$  solutions. However, a similar result would not suffice in the fourth (or higher) order case and the problem must be approached in a different manner. The explicit form of the Green's function is established and used in the invariance problem. The definition of the Green's function is given in Definition 1.6. An argument that the Green's function of Theorem 3.8 satisfies Definition 1.6 may be found in Section 7 of [29].

**THEOREM 3.8:** Let  $b$  be a positive real number and let  $Ly$  be given by (3.1). Then for  $\lambda$  not an eigenvalue of  $Ly$ , the Green's function is given by

$$G(x, z, b) = \begin{cases} \sum_{i=1}^2 \psi_i(z) \phi_i(x), & 0 \leq x < z, \\ \sum_{i=1}^2 \phi_i(z) \psi_i(x), & z \leq x \leq b, \end{cases}$$

where the  $\phi_i$  and  $\psi_i$  are given in Theorem 3.4. Furthermore, for  $f$  in

$L^2(0,b)$ , the function  $\bar{\Phi}$  given by

$$\bar{\Phi}(x) = \int_0^b G(x,z,b)f(z) dz, \quad 0 \leq x \leq b,$$

is a solution of the nonhomogenous differential equation  $Ly = \lambda y - f$ ,  $0 \leq x \leq b$ .

It is noted that the coefficients  $\ell_{rs}(\lambda, b)$  in (3.22) are analytic functions of  $\lambda$  in each of the half-planes  $\text{Im } \lambda > 0$  and  $\text{Im } \lambda < 0$ . For a compact subset  $K$  of one of these half-planes, the monotonic and bounded nature of the surfaces  $S_r(\lambda, b)$  implies, by the Helley selection principle, that there is a strictly increasing sequence of positive real numbers  $\{b_j\}$  such that  $b_j \rightarrow \infty$  and

$$\lim_{j \rightarrow \infty} \ell_{rs}(\lambda, b_j) = m_{rs}(\lambda),$$

and the point  $(m_{r1}(\lambda), m_{r2}(\lambda))$  is on the surface  $S_r(\lambda, \infty)$ . Then by Vitali's theorem on bounded convergence, for example Ash [1:p. 167], the limit process is uniform in  $\lambda$ . Therefore, since each half-plane is simply connected, the limit process is valid throughout each of the half-planes and the limit functions  $m_{rs}(\lambda)$  are analytic in the set  $\text{Im } \lambda \neq 0$ . For  $r = 1$  and  $2$ ,  $\text{Im } \lambda \neq 0$ , define

$$\psi_r(x, \lambda) = \theta_r(x, \lambda) + \sum_{s=1}^2 m_{rs}(\lambda) \phi_s(x, \lambda) \quad (3.24)$$

where the  $\theta_r$  and  $\phi_r$  are as in Theorem 3.1. Note that each  $\psi_r$  is a solution of  $Ly = \lambda y$  and lies in  $L^2(0, \infty)$ .

Before proving the theorem on the invariance of the number of  $L^2(0, \infty)$  solutions, an inequality is required. This inequality is established in Section 7 of [32].

**LEMMA 3.10:** Let  $\lambda$  be a complex number,  $\text{Im } \lambda \neq 0$ . For each  $b > 0$ , let  $G(x, z, b)$  be the Green's function constructed in Theorem 3.8. Let  $\{b_j\}$ ,  $j \geq 1$ , be an increasing sequence of positive real numbers with no finite limit point such that for  $r = 1$  and  $r = 2$ ,

$$\lim_{j \rightarrow \infty} \mathcal{V}_r(x, b_j) = \mathcal{V}_r(x)$$

where the  $\mathcal{V}_r(x, b_j)$  and  $\mathcal{V}_r(x)$  are given in (3.22) and (3.24), respectively. Let  $G(x, z)$  be given by

$$\lim_{j \rightarrow \infty} G(x, z, b_j) = G(x, z).$$

Let  $f$  be in  $L^2(0, \infty)$  and define  $\bar{\Phi}(x)$  by

$$\bar{\Phi}(x) = \int_0^\infty G(x, z) f(z) dz.$$

Then,

$$\int_0^\infty |\bar{\Phi}|^2 dx \leq (1/\text{Im } \lambda)^2 \int_0^\infty |f|^2 dx. \quad (3.25)$$

Note that if  $\{b_j\}$  is a sequence such that as  $j \rightarrow \infty$ , then  $b_j \rightarrow \infty$  and  $G(x, z, b_j) \rightarrow G(x, z)$ . Thus, for  $f$  in  $L^2(0, \infty)$ , (3.25) implies that the function  $\bar{\Phi}$  given by

$$\bar{\Phi}(x) = \int_0^\infty G(x, z) f(z) dz, \quad 0 \leq x < \infty$$

is in  $L^2(0, \infty)$ . Also, the use of elementary methods will show that for  $0 \leq x < \infty$ ,  $L\bar{\Phi} = \lambda\bar{\Phi} - f$  since the continuity of  $G(x, z)$  and its first two quasi-derivatives with respect to  $x$  and the jump discontinuity of  $G(x, z)$  at  $x = z$  are preserved under the uniform limit. It is now possible to show that the number of  $L^2(0, \infty)$  solutions of  $Ly = \lambda y$  is dependent only upon the coefficients  $r$ ,  $p$ , and  $q$  and not upon the

choice of  $\lambda$ , provided only that  $\text{Im } \lambda \neq 0$ . It is this theorem that is the extension of Theorem 2.2 to the fourth-order case. The proof of this result was established by Everitt [28].

THEOREM 3.9: The maximum number of linearly independent solutions of the differential equation  $Ly = \lambda y$  which are in  $L^2(0, \infty)$  is independent of the choice of  $\lambda$ , provided  $\text{Im } \lambda \neq 0$ .

Theorem 3.9 makes the terms limit- $p$ ,  $2 \leq p \leq 4$ , well-defined for  $\text{Im } \lambda \neq 0$ . Thus, the expression (3.1) may be called limit- $p$ ,  $2 \leq p \leq 4$ , depending upon the number of  $L^2(0, \infty)$  solutions there are to the differential equation  $Ly = \lambda y$ . As noted before, the limit-2 case is called the limit-point case and the limit-4 case is called the limit-circle case.



## CHAPTER IV

### EIGENFUNCTION EXPANSIONS

The convergence theory of eigenfunctions associated with singular ordinary differential equations may be considered either in the underlying Hilbert space of square-integrable functions, i.e., the convergence-in-mean theory, or in the classical sense in the space of real or complex numbers, i.e., the direct convergence theory. The convergence-in-mean theory is discussed in Chapters 9 and 10 of the book by Coddington and Levinson [7] and in Naimark's book [65]. The direct convergence theory was developed by many authors, but the more significant contributions in the second-order case were made by E. C. Titchmarsh. His work in final form may be found in his book [74]. The theory has been extended by J. Chaudhuri and W. N. Everitt [6] to formally self-adjoint differential equations of higher order. This extension was via the singular fourth-order problem discussed in Chapter III. The extension of the theory from second to fourth-order involves problems not involved in the second-order case. However, the further extension of the theory from the fourth-order case to any even order expression is largely a matter of notation.

The fourth-order case with one singular endpoint will be considered, as in Chapter III, to keep the notation as simple as possible while allowing sufficient generality to permit extension to higher even order problems with a suitable change of notation. A further

restriction on the coefficients of the expression  $Ly$  will be made in this development in order to establish a lemma. This restriction is in the differentiability requirements to be imposed upon the coefficients of the differential expression. Until that point, as in Chapter III, it is only required that the coefficients be continuous on the half-line  $0 \leq x < \infty$ . Reference will be made to certain results of Chapter III and the notation and definitions of Chapter III will be assumed, that is, the functions  $\phi_r$ ,  $\chi_r$ ,  $\psi_r$ , and  $\bar{\Phi}$  will be as before.

The method of obtaining the eigenfunction expansion is to consider

$$\bar{\Phi}(x, \lambda, f) = \int_0^{\infty} G(x, y, \lambda) f(y) dy$$

as a function of  $\lambda$  for a fixed  $x$  where the function  $f$  is the one to be expanded. It was established in Chapter III that  $\bar{\Phi}(x, \lambda, f)$ , for fixed  $f$  and  $x$ , is an analytic function of  $\lambda$  in the half-planes  $\text{Im } \lambda > 0$  and  $\text{Im } \lambda < 0$ . The function  $\bar{\Phi}$  will be integrated with respect to  $\lambda$  around a large contour, the rectangle defined by the four points  $\pm R + i$ ,  $\pm R + i\delta$ . Then by taking  $R \rightarrow \infty$  and  $\delta \rightarrow 0$  from the right, it will be shown that for certain functions  $F_r(u)$ ,  $r = 1, 2$ , of bounded variation on  $-\infty < x < \infty$ , if  $f$  satisfies certain conditions, then  $f$  has the expansion

$$f(x) = (1/\pi) \sum_{r=1}^2 \lim_{R \rightarrow \infty} \int_{-R}^R \phi_r(x, u) dF_r(u) \quad (4.1)$$

where the integral on the right is a Riemann-Stieltjes integral and for  $r = 1$  and  $2$ ,

$$\phi_r(x, u) = \lim_{v \rightarrow 0^+} \phi_r(x, u + iv).$$

The majority of the development is taken up with a series of lemmas that evaluate the integral of  $\bar{\Phi}$  around the contour and take the limits as  $R \rightarrow \infty$  and  $\delta \rightarrow 0$  from the right. These lemmas are generally very computational.

It is first shown that the problem  $Ly = \lambda y$  with the boundary conditions introduced in Chapter III satisfies the inner product identity

$$(Lu, v) = (u, Lv). \quad (4.2)$$

that is, the problem is self-adjoint.

LEMMA 4.1: Let  $b$  be a positive real number and  $\lambda$  a complex number.

Let  $\phi_r(x, \lambda)$ ,  $\chi_r(x, \lambda, b)$ ,  $r = 1, 2$ , be four solutions of the differential equation  $Ly = \lambda y$ ,  $0 \leq x \leq b$ , with  $Ly$  given in (3.1), such that  $\phi_1$  and  $\phi_2$  take constant real-valued initial conditions at  $x = 0$  in such a way that

$$[\phi_1 \bar{\phi}_2] = 0. \quad (4.3)$$

Similarly,  $\chi_1$  and  $\chi_2$  take constant real-valued initial conditions at  $x = b$  in such a way that

$$[\chi_1 \bar{\chi}_2] = 0. \quad (4.4)$$

In both cases, the initial conditions are independent of  $\lambda$ . Then the problem  $Ly = \lambda y$  with boundary conditions given by

$$[\phi_r y](0) = 0, [\chi_r y](b) = 0, r = 1, 2, \quad (4.5)$$

is self-adjoint, that is, (4.2) holds for  $u$  and  $v$  two functions satisfying (4.5) and having continuous fourth-order quasi-derivatives on  $(0, b)$  satisfying (4.5).

PROOF: Let  $u(x)$  and  $v(x)$  be two functions that have continuous quasi-derivatives with respect to (3.1) up through the fourth order on the interval  $0 \leq x \leq b$ . Then  $u$ ,  $v$ ,  $Lu$ , and  $Lv$  are square-integrable on that interval. By Theorem 1.3,

$$\int_0^b (Lu)\bar{v} - u(L\bar{v}) \, dx = [u \, v](b) - [u \, v](0). \quad (4.6)$$

But, by the definition of the inner product, the left side of (4.6) is the expression  $(Lu, v) - (u, Lv)$ . Thus, it is sufficient to show the right side of (4.6) is zero.

Since the boundary value functions take real values at  $x = 0$ , (4.3) implies

$$[\phi_1 \, \phi_2](0) = 0. \quad (4.7)$$

In Lemma 3.2, let  $n = 2$  and make the substitutions

$$f_1 = u, \, g_1 = v, \, f_2 = g_2 = \phi_1, \, f_3 = g_3 = \phi_2.$$

Then Lemma 3.2 implies

$$\det_{1 \leq i, j \leq 3} ([f_i \, g_j](x)) = 0.$$

for all  $x$ . In particular, for  $x = 0$ , the use of (4.5) and (4.7) yields  $[u \, v](0) = 0$ . In a similar fashion, with  $f_2 = g_2 = \chi_1$  and  $f_3 = g_3 = \chi_2$ , it is seen that  $[u \, v](b) = 0$ . Thus, the right side of (4.6) is zero and the proof of the lemma is complete.

Until otherwise stated, the function  $f(x)$  to be expanded will be assumed to be real-valued. This restriction will be lifted in due course. By Theorem 3.3, the eigenvalues of  $Ly = \lambda y$ ,  $0 \leq x \leq b < \infty$ , are real and the eigenfunctions associated with these eigenvalues must

satisfy real-valued boundary conditions. Thus, without loss of generality, it will be assumed the eigenfunctions are real-valued. The following lemma and its proof may be found in Coddington and Levinson [7:pp. 197-198]. This lemma is the classical Sturm-Liouville expansion theorem for  $2n$ -th order self-adjoint problem on a finite interval with both endpoints regular and will be used to establish other results.

**LEMMA 4.2:** Let  $f$  have continuous quasi-derivatives up through the fourth-order on the compact interval  $0 \leq x \leq b$  and suppose  $f$  satisfies the boundary conditions (4.5). Then on this interval

$$f(x) = \sum_{k=0}^{\infty} (f, y_k) y_k(x) \quad (4.8)$$

where the series converges uniformly on  $0 \leq x \leq b$  and the functions  $y_k$  are the normalized eigenfunctions of the problem  $Ly = \lambda y$  with the conditions (4.3), (4.4), and (4.5). Furthermore, the Parseval equality holds:

$$\int_0^b |f(x)|^2 dx = \sum_{k=0}^{\infty} |(f, y_k)|^2.$$

In the remainder of this chapter, the following functions and definitions will be used. These were introduced in Chapter III. The functions  $G(x, z, \lambda)$  and  $G(x, z, \lambda, b)$  are the Green's functions of Lemma 3.10 and Theorem 3.8. The functions  $\phi_1$ ,  $\phi_2$ ,  $\chi_1$ , and  $\chi_2$  are boundary value functions and the functions  $\psi_1$  and  $\psi_2$  are the two  $L^2(0, \infty)$  solutions of  $Ly = \lambda y$  constructed in Chapter III. The sets  $D(b)$ ,  $0 < b \leq \infty$ , are defined as follows:

**DEFINITION 4.1:** For  $0 < b \leq \infty$ , the function  $f$  is in  $D(b)$  if and only if

- (i)  $f$  is in  $L^2(0, b)$ ,
- (ii)  $f^{[3]}$  is absolutely continuous on each compact subinterval

of  $(0, b)$ ,

(iii)  $Lf$  is in  $L^2(0, b)$ ,

(iv)  $[\phi_r f](0) = 0$  for  $r = 1$  and  $2$ , and

(v)  $\lim_{X \rightarrow b} [\psi_r(\cdot, \lambda) f](X) = 0$  for  $r = 1$  and  $2$  and for all  $\lambda$ .

The functions  $y_k(x, b)$ ,  $k \geq 0$ , will denote the normalized eigenfunctions associated with the eigenvalues  $\lambda_k(b)$  of the eigenvalue problem  $Ly = \lambda y$ ,  $0 \leq x \leq b < \infty$ , with conditions (4.3), (4.4), and (4.5). For convenience, the eigenvalue problem just described will be denoted as the problem  $\pi(b)$ . For  $f$  in  $L^2(0, \infty)$ , the functions  $\Phi$  will be given by

$$\Phi(x, \lambda, b, f) = \int_0^b G(x, z, \lambda, b) f(z) dz, \quad (4.9)$$

$$\Phi(x, \lambda, f) = \int_0^\infty G(x, z, \lambda) f(z) dz. \quad (4.10)$$

A number of properties of the functions in (4.9) and (4.10) will now be established. These are necessary to determine the values of the integrals of  $\Phi$  around the contour mentioned before as  $R \rightarrow \infty$  and  $\delta \rightarrow 0$  from the right. The first lemma establishes the Sturm-Liouville expansion for the function (4.9). The proof is easily established by calculating the coefficients of the expression (4.8).

**LEMMA 4.3:** For  $b > 0$ ,  $f$  a real-valued member of  $D(b)$ , and  $\lambda_k(b)$  the eigenvalues with associated eigenfunctions  $y_k(x, b)$  of the problem  $\pi(b)$ .

$$\Phi(x, \lambda, b, f) = \sum_{k=1}^{\infty} \frac{c_k(b) y_k(x, b)}{\lambda - \lambda_k(b)}$$

$$c_k(b) = \int_0^b f(x) \bar{y}_k(x, b) dx$$

for  $\lambda$  not an eigenvalue of  $\pi(b)$ .

The next lemma gives an interesting relationship between the functions  $\Phi(x, \lambda, f)$  and  $\Phi(x, \lambda, Lf)$  that will prove useful in establishing later results. The proof follows the lines of Section 2.6 of [74].

LEMMA 4.4: For all  $\lambda$ ,  $\text{Im } \lambda \neq 0$  and  $f$  in  $D(\infty)$ ,

$$\lambda \Phi(x, \lambda, f) = f(x) + \Phi(x, \lambda, Lf)$$

for each  $x$ ,  $0 \leq x < \infty$ .

The following estimate of the function  $\Phi(x, \lambda, f)$  will be necessary. This lemma and those results that follow from this lemma are the only results in this development that require certain differentiability conditions on the coefficients of the expression  $Ly$ . The proof is easily established by following the method used for the second-order case in Section 2.14 of [74].

LEMMA 4.5: Let  $Ly$  be given by (3.1) and assume  $r$  is in  $C^2(0, \infty)$ ,  $p$  is in  $C^1(0, \infty)$ , and  $f$  is in  $L^2(0, \infty)$ . Then for  $x$  fixed,  $v \neq 0$ , and  $|\lambda| \geq 1$ ,

$$\Phi(x, \lambda, f) = O(|\lambda|^{1/8} |v|^{-1}). \quad (4.11)$$

In the preceding lemma, the corresponding result for the  $2n$ -th order problem would have  $2^{-n-1}$  as the exponent on  $|\lambda|$  instead of  $1/8$  on the right side of (4.11). The next two lemmas and theorem taken together establish that the function  $\text{Im } \Phi(x, \lambda, f)$  is integrable with respect to  $\lambda$  on any line parallel to and distinct from the real axis. The result is necessary to aid in the integration of  $\Phi$  around the contour in the complex plane. It is first shown the integral exists on the segment  $-1 \leq u \leq 1$ ,  $v = \text{constant}$ ,  $v \neq 0$ . The remainder of the line

$v = \text{constant} \neq 0$  is taken care of in the succeeding lemmas and theorem.

Lemma 4.6 is established in Section 7 of [6].

LEMMA 4.6: Let  $\lambda = u + iv$ ,  $v > 0$ . Then there exists a constant  $K(x)$  depending only upon  $x$  such that

$$\int_{-1}^1 |\operatorname{Im} \Phi(x, \lambda, f)| du \leq K(x) < \infty.$$

Lemma 4.6 is established by using the expansion of  $\Phi(x, \lambda, b, f)$  given in Lemma 4.3 and noting that

$$\operatorname{Im} \Phi(x, \lambda, b, f) = - \sum_{k=0}^{\infty} \frac{v c_k(b) y_k(x, b)}{(u - \lambda_k(b))^2 + v^2}. \quad (4.12)$$

The right side of (4.12) is then integrated between  $-1$  and  $+1$  and estimated using the Schwarz inequality.

The next lemma is also established in Section 7 of [6]. The method of proof is similar to that of the preceding lemma.

LEMMA 4.7: Let  $g$  be a real-valued member of  $D(\infty)$  and let  $\operatorname{Im} \lambda \neq 0$ . Then there exists a constant  $K(x)$ , depending only upon  $x$ , such that for  $\lambda = u + iv$ ,

$$\left\{ \int_{-\infty}^{-1} + \int_1^{\infty} \right\} \left| \frac{\operatorname{Im} \Phi(x, \lambda, g)}{\lambda} \right| du \leq K(x) < \infty.$$

The next theorem makes use of the preceding two lemmas to establish that, as a function of  $\lambda$ ,  $\operatorname{Im} \Phi(x, \lambda, f)$  is integrable on any line parallel to the real axis, but distinct from the real axis. This result is necessary in order to evaluate the integral of  $\Phi$  around the contour mentioned before.



**THEOREM 4.1:** Let  $f$  be a real-valued member of  $D(\infty)$ . Then there exists a parameter  $K(x)$  depending only upon  $x$  such that for  $\lambda = u + iv$ ,  $v > 0$ ,

$$\int_{u_1}^{u_2} |\operatorname{Im} \Phi(x, \lambda, f)| \, du \leq K(x) < \infty \quad (4.13)$$

for all  $u_1$  and  $u_2$  real.

**PROOF:** Since the integrand in (4.13) is nonnegative for all  $u$ , it will suffice to show

$$\int_{-\infty}^{\infty} |\operatorname{Im} \Phi(x, \lambda, f)| \, du \leq K(x) < \infty \quad (4.14)$$

for some parameter  $K(x)$ . From Lemma 4.6, there is a parameter  $K_1(x)$  such that

$$\int_{-1}^1 |\operatorname{Im} \Phi(x, \lambda, f)| \, du \leq K_1(x) < \infty. \quad (4.15)$$

From Lemma 4.4,

$$|\operatorname{Im} \Phi(x, \lambda, f)| \leq |\operatorname{Im} (f(x)/\lambda)| + |\operatorname{Im} (\Phi(x, \lambda, Lf)/\lambda)|.$$

Furthermore, from Lemma 4.7, there is a parameter  $K_2(x)$  such that

$$\left\{ \int_{-\infty}^{-1} + \int_1^{\infty} \right\} \left| \operatorname{Im} \frac{\Phi(x, \lambda, Lf)}{\lambda} \right| \, du \leq K_2(x) < \infty \quad (4.16)$$

Since  $\operatorname{Im} (1/\lambda) = v/(u^2 + v^2)$ , it follows that

$$\left\{ \int_{-\infty}^{-1} + \int_1^{\infty} \right\} \left| \operatorname{Im} \frac{f(x)}{\lambda} \right| \, du = f(x) \left\{ \int_{-\infty}^{-1} + \int_1^{\infty} \right\} \frac{v}{u^2 + v^2} \, du \leq \pi f(x). \quad (4.17)$$

Therefore, there exists a parameter  $K_3(x)$  depending only upon  $x$  such that the integral on the left in (4.17) is bounded by  $K_3(x)$ . By

combining the results (4.15), (4.16), and (4.17), the result (4.14) follows and the proof is complete.

The following lemma is similar in nature to the last theorem and is used later in integrating the function  $\Phi$ . The lemma is established in Section 8 of [6].

**LEMMA 4.8:** For  $-\infty < u_1 < u_2 < \infty$ ,  $\lambda = u + iv$ ,  $0 < v \leq 1$ , and  $1 \leq r, s \leq 2$ , there is a constant  $K(u_1, u_2)$  depending only upon  $u_1$  and  $u_2$  such that

$$\int_{u_1}^{u_2} \operatorname{Im} m_{rs}(u + iv) du \leq K(u_1, u_2)$$

where  $m_{rs}(\lambda)$ ,  $1 \leq r, s \leq 2$ , are the coefficients of the  $L^2(0, \infty)$  solutions  $\mathcal{Y}_r(x, \lambda)$  of the differential equation  $Ly = \lambda y$  defined in (3.24).

The following lemma will be used to construct a set of functions  $k_{rs}$  that are of bounded variation. These functions will in turn be used to construct the functions  $F_r$  that are used in the expansion (4.1) of the function  $f(x)$ . Recall that the coefficients  $m_{rs}(\lambda)$  in the  $L^2(0, \infty)$  solutions of  $Ly = \lambda y$  are analytic in each of the half-planes  $\operatorname{Im} \lambda > 0$  and  $\operatorname{Im} \lambda < 0$ . This lemma states that for almost all real numbers  $u$ ,  $m_{rs}(\lambda)$  is integrable on any rectifiable path lying within one of these half-planes with one endpoint at  $u$ . To establish this, it is sufficient to show that for almost all  $u$ ,  $m_{rs}(\lambda)$  is integrable on  $\lambda = u + iv$ ,  $0 \leq v \leq 1$ . This lemma is proved in Section 9 of [6] and is necessary to arrive at a set of functions of bounded variation defined in the succeeding theorem.

**LEMMA 4.9:** For  $1 \leq r, s \leq 2$ , let  $m_{rs}(\lambda)$  be the coefficients in the

definition of the  $L^2(0, \infty)$  solutions  $\psi_r(x, \lambda)$  of  $Ly = \lambda y$ ,  $\text{Im } \lambda > 0$  given in (3.24). Let  $R > 0$  be arbitrary. Then for almost all  $u$  in  $-R \leq u \leq R$  and for  $1 \leq r, s \leq 2$ ,

$$\int_0^1 |m_{rs}(u + iv)| dv$$

exists and is finite.

The bounded variation functions  $k_{rs}$  that lead to the functions  $F_r$  in the expansion theorem will now be constructed. The proof of this theorem uses the method of contour integration of analytic functions. The theorem follows from Theorem 22.23 of [75].

THEOREM 4.2: For  $1 \leq r, s \leq 2$ , the functions

$$k_{rs}(u) = - \lim_{v \rightarrow 0+} \int_0^u \text{Im } m_{rs}(\mu + iv) d\mu$$

exist for all real  $u$ , are of bounded variation on compact intervals, and satisfy the relations

$$k_{rs}(u) = (1/2)(k_{rs}(u + 0) + k_{rs}(u - 0)), \quad k_{12}(u) = k_{21}(u).$$

Also, the  $2 \times 2$  hermitian matrix

$$[ k_{rs}(u) ], \quad 1 \leq r, s \leq 2$$

is nondecreasing for increasing  $u$ , that is, for  $u_1 \leq u_2$ , the matrix

$$[ k_{rs}(u_2) - k_{rs}(u_1) ]$$

is positive definite or positive semi-definite.

The next two lemmas define the functions  $F_r$  of bounded variation

that are used in the expansion of the function  $f(x)$ . That these functions are of bounded variation will be delayed until Lemma 4.13 since another result will first be necessary. The proofs of the following two lemmas follow the lines of the corresponding second-order results in Sections 3.3 and 3.4 of [74].

LEMMA 4.10: For  $r = 1$  and  $2$  and  $\text{Im } \lambda > 0$ , let  $\Psi_r(x, \lambda)$  be the  $L^2(0, \infty)$  solutions constructed in Theorem 3.6. Then for all  $u$ ,

$$\lim_{v \rightarrow 0+} \int_0^u \text{Im } \Psi_r(x, u + iv) dx = - \int_0^u \sum_{s=1}^2 \phi_s(x, u) dk_{rs}(u). \quad (4.18)$$

LEMMA 4.11: For  $r = 1$  and  $2$ ,  $0 \leq x < \infty$ ,  $-\infty < u < \infty$ , define

$$G_r(x, u) = \sum_{s=1}^2 \int_0^u \phi_s(x, u) dk_{rs}(u). \quad (4.19)$$

Then for each  $u$ ,  $G_r(\cdot, u)$  is in  $L^2(0, \infty)$  and

$$\int_0^\infty G_r^2(x, u) dx$$

is a uniformly bounded function of  $u$ . Furthermore, for  $f$  a real-valued function in  $D(\infty)$ , the function defined by

$$F_r(u) = \int_0^\infty G_r(x, u) f(x) dx \quad (4.20)$$

is finite for all real  $u$ .

The following lemma is used in order to show the functions  $F_r$  defined in the previous lemma are of bounded variation on  $-\infty < u < \infty$ . The details follow the lines of Lemmas 4.6 and 4.7 and Theorem 4.3.

LEMMA 4.12: Let  $\text{Im } \lambda > 0$  and let  $f$  be a real-valued member of  $D(\infty)$ .

Then for  $0 \leq j \leq 3$ , there is a parameter  $K$  such that

$$\int_{-\infty}^{\infty} |\operatorname{Im} \Phi^{[j]}(0, \lambda, f)| du \leq K < \infty. \quad (4.21)$$

**LEMMA 4.13:** Let  $F_r(u)$  be as defined in (4.20). Then for  $r = 1$  or  $2$ ,  $F_r(u)$  is of bounded variation on  $-\infty < u < \infty$ , that is, the total variation of  $F_r(u)$  on  $-R \leq u \leq R$  is bounded independently of  $R$ .

**PROOF:** Let  $\Phi(x, \lambda, f)$  be as defined in (4.10) with  $f$  a real-valued function in  $D(\infty)$ . Then by the comments preceding Theorem 3.9,  $\Phi(x, \lambda, f)$  is a nontrivial solution of the differential equation  $Ly = \lambda y - f$ , and so for at least one  $i$ ,  $0 \leq i \leq 3$ ,  $\Phi^{[i]}(0, \lambda, f)$  is not zero. Without loss of generality, let  $r = 1$ . Since  $\phi_1(x, \lambda)$  is a nontrivial solution of  $Ly = \lambda y$ , for some  $i_0$ ,  $0 \leq i_0 \leq 3$ ,  $\phi_1^{[i_0]}(0, \lambda)$  is not zero. Let

$$K_0 = \max_{1 \leq r \leq 2} |\phi_r^{[i_0]}(0, \lambda)|$$

and recall from the initial conditions satisfied by  $\phi_r$  at  $x = 0$ ,  $K_0$  does not depend on  $\lambda$ . Let  $\{u_j\}$ ,  $0 \leq j \leq k$  be a finite sequence of real numbers such that  $u_{j+1} > u_j$ . Then from (4.21)

$$\int_{-\infty}^{\infty} \left| \sum_{r=1}^2 \phi_r^{[i_0]}(0, \lambda) \operatorname{Im} \int_0^{\infty} \psi_r(z, \lambda) f(z) dz \right| du \leq K.$$

Then since  $0 < |\phi_1^{[i_0]}(0, \lambda)| \leq K_0$ , there is a constant  $K'$  such that for  $r = 1$ ,

$$\int_{-\infty}^{\infty} \left| \operatorname{Im} \int_0^{\infty} \psi_1(z, \lambda) f(z) dz \right| du \leq K'/K_0 = K_1.$$

Then, in particular,

$$\sum_{j=0}^{k-1} \int_{u_j}^{u_{j+1}} \left| \operatorname{Im} \int_0^{\infty} \psi_1(z, \lambda) f(z) dz \right| du \leq K_1. \quad (4.22)$$

But, by the use of the triangle inequality and the Fubini theorem,

(4.22) implies

$$\sum_{j=0}^{k-1} \left| \int_0^{\infty} \left\{ \int_{u_j}^{u_{j+1}} \operatorname{Im} \psi_1(z, \lambda) du \right\} f(z) dz \right| \leq K_1. \quad (4.23)$$

Let  $v \rightarrow 0$  from the right in (4.23) and use (4.18) and (4.19) to obtain

$$\sum_{j=0}^{k-1} \left| \int_0^{\infty} \{ G_1(x, u_{j+1}) - G_1(x, u_j) \} f(z) dz \right| \leq K_1.$$

Then, from (4.20),

$$\sum_{j=0}^{k-1} |F_1(u_{j+1}) - F_1(u_j)| \leq K_1.$$

Since this bound is independent of the range of  $u$ ,  $F_1(u)$  is of bounded variation on  $-\infty < u < \infty$ . The argument for  $F_2(u)$  is entirely similar. This completes the proof of the lemma.

The following three lemmas will perform the integration of  $\operatorname{Im} \Phi(x, \lambda, f)$  with respect to  $\lambda$  around the contour defined by the rectangle in the complex plane with vertices  $\pm R + i$  and  $\pm R + i\delta$ ,  $0 < \delta < 1$ . Then  $R$  will be taken to infinity (through real values) and  $\delta$  to zero. These integrations, after limits are taken, will define an eigenfunction expansion similar to a Fourier integral expansion. In general, the integration along the line  $\operatorname{Im} \lambda = \delta$  is made and the limit taken as  $\delta \rightarrow 0$  from the right and this yields the expansion on the right side of (4.1) and the limits of the integrals along the vertical segments of the rectangle will be zero. The integration along the line  $\operatorname{Im} \lambda = 1$  will yield the left side of (4.1). The first lemma is established in Section 11 of [6].

**LEMMA 4.14:** Let  $f$  be a real-valued function in  $D(\infty)$  and let  $\Phi(x, \lambda, f)$  be as defined in (4.10). Let  $R$  and  $\delta$  be real numbers with  $R > 0$  and  $0 < \delta < 1$ . Then

$$\begin{aligned} \lim_{\delta \rightarrow 0+} \operatorname{Im} \left\{ - (1/\pi) \int_{-R+1}^{R+1} \Phi(x, \lambda, f) d\lambda \right\} \\ = (1/\pi) \sum_{r=1}^2 \int_{-R}^R \phi_r(x, u) dF_r(u). \end{aligned}$$

The next lemma is an easy extension to the fourth-order case of the methods and results given in Section 3.6 of [74].

**LEMMA 4.15:** Let  $f$  be a real-valued function in  $D(\infty)$  and let  $x$  be fixed. Then for real values  $R$ ,

$$\lim_{\substack{\delta \rightarrow 0+ \\ R \rightarrow \infty}} \left\{ \operatorname{Im} \int_{R+1}^{R+1} \Phi(x, \lambda, f) d\lambda \right\} = 0,$$

and a corresponding result holds for  $R \rightarrow -\infty$ . Furthermore,

$$\lim_{R \rightarrow \infty} \int_{-R+1}^{R+1} \Phi(x, \lambda, f) d\lambda = -i\pi f(x)$$

where the integration is taken on the segment joining  $-R+1$  to  $R+1$ .

It is now possible to state and prove the expansion theorem for functions in  $D(\infty)$ . This expansion will be similar to a Fourier integral expansion, the eigenfunctions in this case being those of the eigenvalue problem described at the beginning of this chapter, with  $L_y$  given as (3.1). As mentioned earlier, it will be necessary to assume certain requirements on the differentiability of the coefficients in (3.1). These requirements were necessary in the construction of the proof of Lemma 4.5. It may be that Lemma 4.5 is true even if these requirements are relaxed to having the coefficients locally integrable on  $0 \leq x < \infty$ . The requirements in those chapters preceding this one were to have these coefficients continuous on  $0 \leq x < \infty$ . It should be noted that the expansion theorem is stated for the interval  $0 \leq x < \infty$ .

with  $Ly$  regular at  $x = 0$ , but the theorem holds for any interval  $a \leq x < b$  or  $a < x \leq b$ , with the open endpoint singular and the closed endpoint regular. Application to intervals of this type only requires a suitable change of variable. The theorem is also applicable to an interval of the form  $a < x < b$  with both endpoints singular. The theorem would be applied in this case by choosing a point  $c$ ,  $a < c < b$  and expanding the function on each of the intervals  $a < x \leq c$  and  $c \leq x < b$  and then combining these results.

**THEOREM 4.3:** Let  $Ly$  be given by (3.1) with the coefficients  $r$ ,  $p$ , and  $q$  satisfying the hypotheses of Lemma 4.5. Let the subset  $D(\infty)$  of  $L^2(0, \infty)$  be defined as in Definition 4.1, and for  $1 \leq r, s \leq 2$ , let the functions  $k_{rs}$  be as defined in Theorem 4.2. For  $-\infty < u < \infty$ ,  $0 \leq x < \infty$ , and  $1 \leq r, s \leq 2$ , define

$$G_r(x, u) = \int_0^u \sum_{s=1}^2 \phi_s(x, u) dk_{rs}(u).$$

Then for each  $r$  and for each  $u$ , as a function of  $x$ ,  $G_r(., u)$  is in  $L^2(0, \infty)$ . If for  $r = 1$  and  $2$ ,  $-\infty < u < \infty$ , and  $f$  in  $D(\infty)$ ,  $F_r(u)$  is defined by

$$F_r(u) = \int_0^\infty G_r(t, u) f(t) dt,$$

then  $F_r$  is of bounded variation on  $-\infty < u < \infty$  and for each  $x \geq 0$

$$f(x) = (1/\pi) \sum_{r=1}^2 \int_{-\infty}^\infty \phi_r(x, u) dF_r(u) \quad (4.24)$$

where

$$\int_{-\infty}^\infty \phi_r(x, u) dF_r(u) = \lim_{b \rightarrow \infty} \int_{-b}^b \phi_r(x, u) dF_r(u).$$



PROOF: First, assume  $f$  is real-valued. The statement concerning the functions  $G_r$  was established in Lemma 4.11. The functions  $F_r$  were shown to be of bounded variation in Lemma 4.13. Statement (4.24) remains to be established. Let  $R$  and  $\delta$  be positive numbers and consider the contour  $C$  defined as the negatively oriented rectangle with vertices  $\pm R + i$  and  $\pm R + i\delta$ . Since the singularities of  $\Phi(x, \lambda, f)$  are all real,

$$(1/\pi) \int_C \Phi(x, \lambda, f) d\lambda = 0,$$

in particular,

$$(1/\pi) \operatorname{Im} \int_C \Phi(x, \lambda, f) d\lambda = 0. \quad (4.25)$$

From Lemma 4.14, for each  $R > 0$  and  $\delta > 0$ ,

$$\lim_{\delta \rightarrow 0+} (1/\pi) \int_{R+i\delta}^{-R+i\delta} \operatorname{Im} \Phi(x, \lambda, f) d\lambda = (1/\pi) \sum_{r=1}^2 \int_{-R}^R \phi_r(x, u) dF_r(u). \quad (4.26)$$

By Theorem 4.1, the integrals in (4.26) converge as  $R \rightarrow \infty$ , thus

$$\lim_{\substack{\delta \rightarrow 0+ \\ R \rightarrow \infty}} \int_{R+i\delta}^{-R+i\delta} \operatorname{Im} \Phi(x, \lambda, f) d\lambda = \sum_{r=1}^2 \int_{-\infty}^{\infty} \phi_r(x, u) dF_r(u). \quad (4.27)$$

From Lemma 4.15,

$$\lim_{R \rightarrow \infty} (1/\pi) \int_{-R+i}^{R+i} \operatorname{Im} \Phi(x, \lambda, f) d\lambda = -f(x), \quad (4.28)$$

and

$$\lim_{\substack{\delta \rightarrow 0+ \\ R \rightarrow \infty}} (1/\pi) \left\{ \int_{R+i\delta}^{R+i} + \int_{-R+i}^{-R+i\delta} \right\} \operatorname{Im} \Phi(x, \lambda, f) d\lambda = 0. \quad (4.29)$$

Thus, by integrating  $\operatorname{Im} \Phi(x, \lambda, f)$  around the contour  $C$  and then letting  $\delta \rightarrow 0+$  and  $R \rightarrow \infty$ , (4.25), (4.27), (4.28), and (4.29) imply for real-valued  $f$ ,

$$f(x) = (1/\pi) \sum_{r=1}^2 \int_{-\infty}^{\infty} \phi_r(x, u) dF_r(u). \quad (4.30)$$

The expansion (4.30) for complex-valued functions in  $D(\infty)$  may be accomplished by writing  $f(x) = \operatorname{Re} f(x) + i \operatorname{Im} f(x)$ , then finding the expansion (4.30) for each of the real and imaginary parts and finally taking the linear combination of the results. This completes the proof of the theorem.

Following is a theorem which indicates the reasoning behind the desire to consider the limit-point case of a singular formally self-adjoint differential expression. The theorem states that, in Definition 4.1, if  $Ly$  is in the limit-point case and if a function  $f$  satisfies conditions (i) through (iv), then the function also satisfies (v). In other words, in the limit-point case, a condition at  $x = \infty$  need not be imposed upon the function to be expanded. A lemma is required prior to proving the theorem. The following lemma is Lemma 3.2 of [35].

**LEMMA 4.16:** Suppose the complex-valued measurable functions  $f$  and  $g$  satisfy the conditions

- (i)  $f$  is in  $L^2(0, \infty)$ ,
- (ii)  $g$  is in  $L^2(a, b)$  for all  $a$  and  $b$ ,  $0 \leq a < b < \infty$ , and
- (iii)  $g$  is not in  $L^2(0, \infty)$ .

Then

$$\lim_{b \rightarrow \infty} \frac{\int_0^b f(x)g(x) dx}{\left\{ \int_0^b |g(x)|^2 dx \right\}^{1/2}} = 0.$$

The following theorem is used extensively in theorems establishing the limit-point case for  $2n$ -th order differential equations. The theorem is stated and proved for the fourth-order case.

THEOREM 4.4: Let  $Ly$  be given by (3.1) and suppose the coefficients  $r$ ,  $p$ , and  $q$  are real-valued and continuous with  $p(x) > 0$  for all  $x \geq 0$ .

Let  $E$  be that subset of  $L^2(0, \infty)$  defined by:  $u$  is in  $E$  if and only if

- (i)  $u$  is in  $L^2(0, \infty)$ ,
- (ii)  $u^{[3]}$  is locally absolutely continuous on  $0 \leq x < \infty$ ,
- (iii)  $Lu$  is in  $L^2(0, \infty)$ , and
- (iv)  $[u \phi_i](0) = 0$  for  $i = 1$  and  $2$ .

Then  $L$  is in the limit-point case if and only if

$$\lim_{b \rightarrow \infty} [u v](b) = 0 \quad (4.31)$$

for each  $u$  and  $v$  in  $E$ .

PROOF: Suppose (4.31) holds for all  $u$  and  $v$  in  $E$ . Suppose further, that  $Ly$  is not in the limit-point case. Then there are complex numbers  $\alpha_1$  and  $\alpha_2$ , not both zero, such that

$$Y(x) = \alpha_1 \phi_1(x, \lambda) + \alpha_2 \phi_2(x, \lambda)$$

is in  $L^2(0, \infty)$ . Then, since each  $\phi_i$  is a solution of  $Ly = \lambda y$ , it follows that  $LY = \lambda Y$ , and thus,  $LY$  is in  $L^2(0, \infty)$ . Clearly, condition (ii) of the definition of  $E$  is satisfied. From Chapter III,

$$[Y \phi_i](0) = 0, \quad i = 1, 2,$$

and so, (iv) is satisfied. Thus,  $Y$  is in  $E$ . Therefore, by Theorem 1.3,

$$[Y Y](b) = [Y Y](0) + 2i \nu \int_0^b |Y(x)|^2 dx. \quad (4.32)$$

Since  $Y$  is a linear combination of  $\phi_1$  and  $\phi_2$ , the first term on the right of (4.32) is zero. By (4.31), the left side is  $o(1)$  as  $b$  tends to

infinity, and thus

$$\lim_{b \rightarrow \infty} \int_0^b |Y(x)|^2 dx = 0,$$

implying  $\alpha_1 = \alpha_2 = 0$ , a contradiction. Thus,  $L_y$  is in the limit-point case.

For the converse statement, suppose  $L$  is in the limit-point case and  $u$  and  $v$  are members of  $E$ . Let  $\{b_k\}$ ,  $k \geq 0$ , be a strictly increasing sequence of positive real numbers with no finite limit point. Let  $G(\emptyset; b_k)$  be defined as in (3.5) and for each  $k$ , let  $r_1(k)$  and  $r_2(k)$  be the characteristic roots of  $G(\emptyset; b_k)$ . By Lemma 3.5,  $G(\emptyset; b_k)$  is positive definite and thus these roots are not zero. For each  $k \geq 0$ , let  $V(k) = (\alpha_{ij}(b_k))$  be the unitary  $2 \times 2$  matrix such that

$$V(k)G(\emptyset; b_k)V^*(k) = \text{diag} [r_1(k), r_2(k)]. \quad (4.33)$$

For  $i = 1$  and  $2$ , define  $y_i(x, k)$  by

$$y_i(x, k) = \alpha_{i1}(b_k)\phi_1(x) + \alpha_{i2}(b_k)\phi_2(x).$$

Then the left side of (4.33) is  $G(y; b_k)$  where  $y$  is the vector given by  $y = (y_1(x, k), y_2(x, k))^T$ . Therefore, by (4.33),

$$\int_0^{b_k} y_i(x, k) \bar{y}_j(x, k) dx = r_i(k) \delta_{ij}$$

for all  $k \geq 0$  and  $1 \leq i, j \leq 2$ . It then follows that since  $[y_i \ y_j](0)$  is zero,

$$[y_i \ y_j](b_k) = r_i(k) \delta_{ij}. \quad (4.34)$$

Also, note (4.3) implies

$$[y_1 \bar{y}_j](b_k) = 0, \quad 1 \leq i, j \leq 2. \quad (4.35)$$

Since the matrix  $V$  is unitary,

$$|\alpha_{i1}(b_k)|^2 + |\alpha_{i2}(b_k)|^2 = 1$$

for  $i = 1$  and  $2$ . Thus, these coefficients are bounded, implying there is an increasing sequence of positive integers  $n$  such that for  $i, j = 1, 2$

$$\alpha_{ij}(b_n) \rightarrow \alpha_{ij} \quad (4.36)$$

and for  $i = 1$  and  $2$ , not both  $\alpha_{i1}$  and  $\alpha_{i2}$  are zero. For  $i = 1$  and  $2$ , define

$$Y_i(x) = \alpha_{i1}\phi_1(x) + \alpha_{i2}\phi_2(x).$$

Then, since not both  $\alpha_{i1}$  and  $\alpha_{i2}$  are zero, neither  $Y_i$  can be in  $L^2(0, \infty)$  since the limit-point case holds, but, clearly both  $Y_1$  and  $Y_2$  are in  $L^2(a, b)$  for all  $a$  and  $b$ ,  $0 \leq a < b < \infty$  by continuity of solutions.

From (4.36),

$$y_1(x, k) = Y_1(x) + o(1).$$

Consider the expression

$$\frac{[(Y_1 + o(1))v](b_k)}{\left\{\int_0^{b_k} |Y_1(x) + o(1)|^2 dx\right\}^{1/2}}. \quad (4.37)$$

By Theorem 1.3, the numerator of (4.37) may be expressed as

$$\int_0^{b_k} (Y_1(x) + o(1))(\lambda \bar{v}(x) - (Lv)(x)) dx$$

since  $v$  is in  $E$ ,  $[Y_k v](0) = 0$ , and  $LY_1 = \lambda Y_1$ .

In Lemma 4.16, let  $f(x) = \lambda \bar{v}(x) - \overline{(Lv)}(x)$  and let  $g(x) = Y_1(x) + o(1)$ . From the definition of  $E$  and the above comments,  $f$  and  $g$  satisfy the hypotheses of Lemma 4.16 and therefore, the expression (4.37) is  $o(1)$  as  $k \rightarrow \infty$ .

Theorem 3.2 will now be applied. Let  $f_1 = \bar{f}_3 = g_1 = \bar{g}_3 = y_1(.,k)$ ,  $f_2 = \bar{f}_4 = g_2 = \bar{g}_4 = y_2(.,k)$ ,  $f_5 = u$ , and  $g_5 = v$ . By (4.34) and (4.35), for each  $k$ , the upper left  $4 \times 4$  submatrix of the matrix in Theorem 3.2 is a diagonal matrix. Divide the first two rows and first two columns by the term

$$\left\{ \int_0^{b_k} |y_1(x,k)|^2 dx \right\}^{1/2} \quad (4.38)$$

for  $i = 1$  and  $2$  respectively. Similarly, divide the second two rows and second two columns by (4.38),  $i = 1$  and  $2$  respectively. Then, it follows that the upper left  $4 \times 4$  submatrix is

$$\text{diag} [2iv, 2iv, -2iv, -2iv].$$

The element in the  $(5,5)$  position is  $[u \ v](b_k)$ . A typical element in the fifth row or fifth column (except the  $(5,5)$  element) is given in (4.37) and thus is  $o(1)$  as  $k$  tends to infinity. Therefore, the determinant of the matrix is given by

$$2^4 v^4 [u \ v](b_k) + o(1). \quad (4.39)$$

Therefore, since by Theorem 3.2, the determinant is identically zero, (4.39) implies  $[u \ v](b_k) \rightarrow 0$  as  $k \rightarrow \infty$ . Since the sequence  $b_k$  is arbitrary, (4.31) then follows. This completes the proof of Theorem 4.4.

It should be noted that condition (iv) is not necessary for the

second half of the proof, for if  $u$  and  $v$  satisfy (i), (ii), and (iii) of the hypothesis, then these functions can always be redefined on  $0 \leq x \leq 1$  so that (iv) holds.

## CHAPTER V

### THE DEFICIENCY INDEX PROBLEM

This chapter will survey the development of the limit-point and limit-circle problem for formally self-adjoint differential expressions of even order with real-valued coefficients defined on an interval with one singular endpoint. No attempt will be made to state and prove all known results for this problem since the list is rather lengthy. Also, many of the results require considerable development of the theory of linear operators on function spaces. In general, three basic techniques for determining the limit-p case of a differential expression will be considered. Two of these methods are applicable to differential expressions of the second and fourth-order, but little has been accomplished with expressions of higher order. The first of these methods is particularly applicable to the second-order problem. The idea here is to establish the existence (or nonexistence) of two linearly independent solutions of  $Ly = \lambda y$  that lie in  $L^2(0, \infty)$ . If this can be accomplished, the second-order problem is completely determined since these are the only two cases. That is, if there is a basis of the solution space lying in  $L^2(0, \infty)$ , the limit-circle case occurs and if one solution of  $Ly = \lambda y$  can be found that is not square-integrable, the limit-point case occurs. This method is less used in the fourth-order case since only a determination can be made of whether the expression is or is not limit-4 (limit-circle) and less information is obtained



since the limit-point (limit-2) or limit-3 case may occur.

The second of the two methods makes use of Theorem 4.4. This theorem is true for the  $2n$ -th order problem by modifying hypothesis (ii) to read that  $u^{[2n-1]}$  is locally absolutely continuous on the interval  $[0, \infty)$  and by running the index  $i$  in (iv) from one to  $n$  where the bilinear form used is with respect to the particular operator under consideration. In this method, a theorem is formulated by putting conditions on the coefficients of the differential expression which will force  $[u \ v](\infty) = 0$  for all  $u$  and  $v$  in  $D(\infty)$ . In this case, the limit-point (limit- $n$ ) case will occur. Similarly, if two functions  $u$  and  $v$  in the subset  $E$  of  $L^2(0, \infty)$  of Theorem 4.4 can be constructed so that  $[u \ v](\infty) \neq 0$ , then the limit-point case does not occur. As before, this completely determines the second-order problem, but not higher order problems. A combination of this method and the first may be applied in the fourth-order problem to eliminate the limit-point and limit-circle cases, leaving the limit-3 case. The two methods so far described are less applicable in the  $2n$ -th order case,  $n > 2$ , since these cannot determine the limit- $p$  case,  $n < p < 2n$ . However, for application to the expansion theorem, it is only necessary to know whether or not the limit- $n$  case occurs in order to determine the necessity for imposing a boundary condition at infinity upon the function to be expanded.

The third method for determining the limit- $p$  case is called the asymptotic method. With the exception of the second-order, and in some cases, the fourth-order expressions, the only known method for identifying the limit- $p$  case,  $p > n$ , is by this asymptotic method. In this method, one attempts to obtain asymptotic estimates for the rates of

growth of a complete set of linearly independent solutions for the equation  $Ly = \lambda y$ ,  $\text{Im } \lambda \neq 0$ . Knowing the rate of growth of the solutions in this basic set, one can usually determine the dimension of the subspace of the solution space that lies in  $L^2(0, \infty)$ . A more complete description of the asymptotic method will be made later, along with an example of its use.

The problem of determining the limit- $p$ ,  $n \leq p \leq 2n$ , case of a linear differential operator is commonly called the deficiency index problem. A complete description of the reasoning behind this name would take considerable development of the general theory of linear differential operators, but an informal description will be given. A complete development may be found in Naimark's book [65]. In the theory of linear differential operators, one may consider the formal operator

$$My = \sum_{k=0}^n (-1)^k (p_{n-k} y^{(k)})^{(k)} \quad (5.1)$$

operating on certain functions that are defined on an interval  $(a, b)$ . The coefficients  $p_k$  are assumed to be real-valued with  $p_0$  taking only positive values. The formal operator (5.1) is self-adjoint in the sense that for any two functions  $u$  and  $v$  having continuous derivatives of all orders and vanishing outside some compact subinterval of  $(a, b)$ , the inner product identity,

$$(Mu, v) = (u, Mv) \quad (5.2)$$

holds. Hence, when restricted to the test functions,  $M$  is a densely defined symmetric operator, say  $L'_0$ , in the Hilbert-Lebesgue space  $L^2(0, \infty)$  and so has a symmetric, closed extension  $L_0$ , called the minimum operator associated with  $M$ . The above statements are discussed in

Section 17.4 of [65]. The operator  $L = L_0^*$ , the adjoint of  $L_0$  described in Chapter I, is called the maximal operator associated with  $M$ .  $L$  is a closed operator which is the restriction of  $M$  to those functions  $u$  such that  $u$  and  $Mu$  are in  $L^2(a,b)$  and  $u^{[2n-1]}$  is locally absolutely continuous on  $(a,b)$ .

The formal self-adjoint operators which arise from physical problems usually come equipped with natural boundary conditions at  $a$  and  $b$ . If these are of the proper type and number, then the restriction of  $L$  to those elements in its domain which satisfy the boundary conditions is a self-adjoint operator  $H$  which satisfies the relation  $L_0 \subseteq H \subseteq L$ . The operator  $L$  of Chapter IV defined on  $D(\infty)$  is such an operator provided the limit-point case occurs. As shown in Chapter IV, such operators allow expansions of functions in  $L^2(a,b)$  which satisfy certain boundary conditions. Recall that no conditions at the singular endpoint(s) are required on the function to be expanded if the differential expression is in the limit-point case. If the expression is not limit-point, some conditions on the function to be expanded must be imposed at the singular endpoint(s). In Chapter IV, the endpoint  $x = 0$  was taken to be regular and the endpoint  $x = \infty$  was taken to be singular. At the regular endpoint, boundary conditions were imposed since the expression can be considered to be limit-2n at that endpoint. That is, for  $b > 0$ , all solutions of  $Ly = \lambda y$  are square-integrable on  $0 \leq x \leq b$ , and so the expression is not self-adjoint on  $0 \leq x \leq b$ . Thus, boundary conditions must be imposed at a regular endpoint. At the singular endpoint,  $x = \infty$ , boundary conditions were needed only if the limit-point case did not occur.

In the theory of the extensions of symmetric operators described

by Naimark and developed by von Neumann, two cardinals are vital. To describe these cardinals, first let  $D(L_0)$  denote the domain of the operator  $L_0$ . Define the range spaces  $R_\lambda$  and  $R_{\bar{\lambda}}$  by

$$R_\lambda = (L_0 - \lambda I)D(L_0), \quad R_{\bar{\lambda}} = (L_0 - \bar{\lambda} I)D(L_0) \quad (5.3)$$

where  $\lambda$  is a complex number,  $\text{Im } \lambda > 0$ . Then define  $N_\lambda$  and  $N_{\bar{\lambda}}$  as the orthogonal complements of  $R_\lambda$  and  $R_{\bar{\lambda}}$ , respectively, in  $L^2(a,b)$ . The deficiency numbers of  $L_0$  are then defined by

$$k = \dim N_\lambda, \quad m = \dim N_{\bar{\lambda}}.$$

The pair  $(k,m)$  is called the deficiency index of  $L_0$ . An application of (5.2) shows that if  $y$  is in  $N_{\bar{\lambda}}$  and if  $z$  is any member of  $D(L_0)$ , then

$$(L_0 - \bar{\lambda} I)(z), y) = 0.$$

That is, by (5.2),

$$(L_0 z - \bar{\lambda} z, y) = 0$$

and therefore, the inner product identity

$$(z, L_0 y) = (z, \lambda y)$$

holds. Since  $z$  is an arbitrary member of  $D(L_0)$ , it follows then that  $y$  is a solution of  $L_0 y = \lambda y$  that lies in  $L^2(a,b)$ . But since  $L_0 = M^* = M$  by (1.6), the deficiency number  $m$  is the dimension of the subspace of the solution space of  $My = \lambda y$  that lies in  $L^2(a,b)$ . From (5.3) and since the coefficients of  $M$  are real-valued, it follows that  $k = m$ . From the above comments, the deficiency index problem is the same problem as determining the maximal number of linearly independent solutions

of  $My = \lambda y$  that lie in  $L^2(a,b)$ . Thus, there is reason to study the deficiency index problem. However, perhaps the primary reason for studying the deficiency index problem is "... because it is difficult, and therefore challenging" [11, p. 355].

The deficiency index problem, as now known, dates back to Hermann Weyl [79], around 1910. Until about the mid 1940's, only a few papers appeared dealing with this problem. At that time, several contributions dealing with the deficiency index problem appeared. Most notable among these is Eigenfunction Expansions by E. C. Titchmarsh [74]. Weyl's comments, made in a Gibbs lecture [80], on the span of nearly forty years without consideration of the problem are interesting. His comments refer to the above mentioned work by Titchmarsh and to a major paper by Kunihiko Kodaira [59].

It is remarkable that forty years had to pass before such a thoroughly satisfactory direct treatment emerged; the fact is a reflection on the degree to which mathematicians during this period got absorbed in abstract generalizations and lost sight of their task of finishing up some of the more concrete problems of undeniable importance (p. 124).

The current thrust of the work on the deficiency index problem is toward determining necessary and sufficient conditions on the coefficients of the differential expression to establish the limit-p case. There have been a large number of sufficient conditions found, but so far, necessary conditions have been elusive. Some of the results for second and fourth-order problems will be discussed. The first series of results will be examples of the first two methods described earlier. Following these, an example of the asymptotic method will be given. The asymptotic method is quite difficult and technical, and as a result, only one example will be given. After this example is given, a remarkable theorem that indicates the delicacy of the problem of finding

necessary conditions will be proved. In all cases that follow, the following notations and conditions will be assumed.

$$L_2 y = - (py')' + qy, \quad (5.4)$$

$$L_4 y = (ry'')'' - (py')' + qy. \quad (5.5)$$

The coefficients  $r$ ,  $p$ , and  $q$  of (5.4) and (5.5) will be assumed to be real-valued and continuous on the interval  $0 \leq x < \infty$  with  $p(x) > 0$  for all  $x$  in (5.4) and  $r(x) > 0$  for all  $x$  in (5.5). Other conditions may be imposed in the individual theorems. In the second-order problem, the functions  $\phi$  and  $\psi$  will be those introduced in Chapter II and in the fourth-order problem, the functions  $\phi_1$ ,  $\phi_2$ ,  $\psi_1$ , and  $\psi_2$  will be those introduced in Chapter III. Recall that  $\psi$ ,  $\psi_1$ , and  $\psi_2$  are all in  $L^2(0, \infty)$ .

The following result is due to Levinson and is one of the more widely known conditions for the limit-point case for second-order differential expressions. His result is in terms of a comparison function for the coefficient  $q(x)$ . This theorem is Theorem 2.4 of Chapter 9 of [7].

**THEOREM 5.1:** Let  $L_2 y$  be given by (5.4). Suppose  $M(x)$  is a positive, differentiable function such that

- (i)  $(pM)^{-1/2}$  is not in  $L(0, \infty)$ , and
- (ii)  $M'p^{1/2}M^{-3/2}$  is bounded.

Suppose further, that for some  $K > 0$ ,

- (iii)  $q(x) > -KM(x)$  eventually.

Then  $L_2 y$  is in the limit-point case.

The following theorem is due to Titchmarsh and is similar to Theorem 5.1. In this theorem, if  $p(x) = 1$  and the bounding function  $M$  is assumed to be nondecreasing, then hypothesis (ii) of Theorem 5.1 is not needed. This theorem is Theorem 2.20 of [74].

THEOREM 5.2: Let  $L_2 y$  be given by (5.4) and assume  $p(x) = 1$  for all  $x \geq 0$ . Suppose  $M(x)$  is a positive, continuous, and nondecreasing function such that  $M^{-1/2}$  is not integrable on  $0 \leq x < \infty$ . Then,  $L_2 y$  is in the limit-point case provided  $q(x) \geq -M(x)$  for all  $x \geq 0$ .

Of historical interest is Weyl's original result from his paper [79]. This result follows immediately from either Theorem 5.1 or 5.2' by setting  $M(x) = K$ .

COROLLARY 5.1: Let  $L_2 y$  be given by (5.4) and assume  $p(x) = 1$  for all  $x \geq 0$ . If for some  $K > 0$ ,  $q(x) \geq -K$  for all  $x \geq 0$ , then  $L_2 y$  is in the limit-point case.

The following result was established independently by Titchmarsh [73] and Hartman and Wintner [51] in 1949. As in the first two theorems, the interest is in the growth of the coefficient  $q$  and this result allows more growth than Weyl's result.

THEOREM 5.3: Let  $L_2 y$  be given by (5.4) and assume  $p(x) = 1$  for all  $x \geq 0$ . If there exists a constant  $K > 0$  such that  $q(x) \geq -Kx^2$  for all  $x \geq 0$ , then  $L_2 y$  is in the limit-point case. Furthermore, the exponent 2 is the best possible in the sense that if 2 is replaced by  $2 + \delta$ ,  $\delta > 0$ , the result may no longer be true.

PROOF: The first part of the theorem follows from Theorem 5.2 by

setting  $M(x) = Kx^2$ . To show the second part, the second-order analogy of Theorem 4.4 will be used.

Let  $L_y$  be given by

$$L_y = -y'' - x^{2+\delta}y, \quad \delta > 0. \quad (5.6)$$

Let the function  $u$  be defined by

$$u(x) = \begin{cases} f(x), & 0 \leq x \leq 1, \\ (x^{2+\delta})^{-1/4} \exp(i \int_0^x (t^{2+\delta})^{1/2} dt), & 1 < x < \infty, \end{cases}$$

where  $f$  is any  $C^1(0,1)$  function such that  $f(0) = f'(0) = 0$  and such that  $u^{[1]}$  is absolutely continuous on  $0 \leq x \leq 2$ . It may then be calculated directly that  $u$  and  $Lu$  are in  $L^2(0, \infty)$  and that  $u^{[1]}$  is locally absolutely continuous on  $0 \leq x < \infty$ . Also, it is easily seen that condition (iv) of Theorem 4.4 is satisfied (for the second-order case.) Now, for the hermitian form  $[u \ v]$  with respect to (5.6), for  $b > 1$ , it may easily be calculated that

$$[u \ u](b) = -2i \operatorname{Im}(\bar{u}(b)u'(b)) = 2.$$

By Theorem 4.4,  $L_y$  is not in the limit-point case, implying  $L_y$  is in the limit-circle case. This completes the proof of the theorem.

The following result is also due to Hartman and Wintner [50]. It is similar in its restrictions on the growth of  $q$ .

**THEOREM 5.4:** Let  $L_2y$  be given by (5.4) and assume  $p(x) = 1$  for all  $x \geq 0$ . If for a certain constant  $c > 0$ , and for some  $K > 0$ , the inequality.

$$q(x_2) - q(x_1) > -K(x_2 - x_1) \quad (5.7)$$



holds for  $c < x_1 < x_2$ , then  $L_2y$  is in the limit-point case.

PROOF: Let  $x_2 = x$  and keep  $x_1$  fixed in (5.7). Then

$$q(x) > -K(x - x_1) + q(x_1).$$

Thus, for  $x$  sufficiently large, for some constant  $K_1$ ,  $q(x) > -K_1x$ .

The result now follows from Theorem 5.2 by taking  $M(x) = K_1x$ , and the proof is complete.

The next result leads to the generalization of Weyl's result in Corollary 5.1 to the more general expression (5.4). In Coddington and Levinson's book [7],  $L_2y$  was shown to be in the limit-point case if  $q$  is bounded below and  $p^{-1/2}$  is not in  $L(0, \infty)$ . This is an immediate corollary of Levinson's theorem, Theorem 5.1, by taking  $M(x) = 1$ . In 1966, Everitt [34] removed the condition that  $p^{-1/2}$  is not in  $L(0, \infty)$ . His proof, however, is tedious. In 1969, Wong [82] gave an elegant proof of the result by using the mean value theorem for derivatives. The result is a corollary of the following theorem which connects the ideas of the limit-point and limit-circle cases to the notion of oscillatory differential equations. A differential equation is said to be oscillatory if the equation has at least one oscillatory solution, that is, has at least one nontrivial solution with an infinite number of zeros. This theorem and its corollary are due to Kurss [61].

THEOREM 5.5: Let  $L_2y$  be given by (5.4) and let a comparison operator be defined by

$$My = -(py')' + q_1y.$$

Then  $L_2y$  is in the limit-point case if

- (i)  $M$  is in the limit-point case and is nonoscillatory, and  
 (ii)  $q - q_1$  is bounded below.

PROOF: By (i), there is a solution  $v$  of  $My = 0$  that is strictly positive for sufficiently large  $x$  and is not in  $L^2(0, \infty)$ . Also, (ii) implies there exists a real number  $\lambda$  such that for  $x$  sufficiently large,  $q - q_1 \geq \lambda$ . Without loss of generality, assume the above holds for all  $x \geq 0$ . Let  $y$  be the solution of  $L_2 y = \lambda y$  that satisfies the initial conditions  $y(0) = v(0)$  and  $y'(0) = v'(0)$ . Then since  $q(x) - \lambda \geq q_1$  and  $(L_2 - \lambda)y = 0$ , the Sturm Comparison Theorem implies  $y(x) \geq v(x)$  for all  $x \geq 0$ . Thus,  $y$  is not in  $L^2(0, \infty)$  and the limit-circle case cannot hold. Therefore,  $L_2 y$  is in the limit-point case and the proof is complete.

COROLLARY 5.2: Let  $L_2 y$  be given by (5.4). Then  $L_2 y$  is in the limit-point case if  $q$  is bounded below.

PROOF: Let  $q_1(x) = 0$  for all  $x \geq 0$ . Then the solutions of  $My = 0$  are linear combinations of the functions

$$u(x) = 1, \quad v(x) = \int_0^x 1/p(t) \, dt$$

and thus do not oscillate. Also,  $My$  cannot be in the limit-circle case since  $u$  is not in  $L^2(0, \infty)$ . Therefore, the result holds by Theorem 5.5. This completes the proof of the corollary.

The following result given a different restriction on the growth of  $q$ . In this theorem, if  $q$  is in  $L^2(0, \infty)$ , then the limit-point case holds. Thus,  $q^-$  may be allowed to be arbitrarily large, but only on very small sets. The theorem in the special case  $p(x) = 1$  is due to

Putnam [69]. To the author's knowledge, the following generalization has not appeared.

**THEOREM 5.6:** Let  $L_2y$  be given by (5.4) and assume  $q$  is in  $L^2(0, \infty)$ . Then  $L_2y$  is in the limit-point case.

**PROOF:** It will be shown that if two solutions of  $L_2y = 0$  are in  $L^2(0, \infty)$ , then they are necessarily linearly dependent, implying the limit-circle case cannot occur.

Let  $y$  be any solution of  $L_2y = 0$  such that  $y$  is in  $L^2(0, \infty)$ . Then

$$(py')'(x) = q(x)y(x). \quad (5.8)$$

Integration of both sides of (5.8) from 0 to  $x$  yields the identity

$$[(py')(t)]_0^x = \int_0^x q(t)y(t) dt \quad (5.9)$$

and it follows by the Schwarz inequality that the right side of (5.9) is bounded as  $x \rightarrow \infty$ . Thus, as  $x \rightarrow \infty$ ,

$$p(x)y'(x) = o(1). \quad (5.10)$$

Let  $y$  and  $z$  be any two solutions of  $L_2y = 0$  that are in  $L^2(0, \infty)$ . Then, since

$$[y \bar{z}](x) = p(x)y(x)z'(x) - p(x)y'(x)z(x), \quad (5.11)$$

(5.10) implies each of the terms on the right side of (5.11) is in  $L^2(0, \infty)$ . Thus,  $[y \bar{z}](x)$  is in  $L^2(0, \infty)$ . By Theorem 1.3,  $[y \bar{z}](x)$  is independent of  $x$ , and so (5.11) must be identically zero. Since (5.11) is the generalized Wronskian for solutions of  $L_2y = 0$ ,  $y$  and  $z$  cannot be linearly independent. Therefore, the limit-circle case cannot

occur, and the conclusion follows. This completes the proof of the theorem.

The following two theorems are due to Wong and Zettl [85]. These theorems also make use of a comparison operator as in Theorem 5.5. The second of these results involves the oscillatory nature of the two operators.  $L_2y$  will be the expression (5.4) with the added condition that  $q(x) < 0$  for  $x \geq 0$ . Define the comparison operator

$$Mz = - (z'/q)' + z/p, \quad (5.12)$$

where  $q$  and  $p$  are the coefficients of  $L_2y$ . Also, assume  $p$  and  $q$  are continuously differentiable on the interval  $0 \leq x < \infty$ . The proofs of these theorems are found in [85].

**THEOREM 5.7:** Let  $Y = (qp)'/(qp)$  and  $Mz$  be as defined in (5.12). If either

- (i)  $Y^+$  is in  $L(0, \infty)$ , or
- (ii)  $Y^-$  is in  $L(0, \infty)$  and  $-qp$  is bounded above,

then  $L_2y$  is in the limit-point case.

The next theorem considers the oscillatory properties of the comparison operator  $Mz$  and of  $L_2y$ .

**THEOREM 5.8:** Suppose  $1/p$  is not in  $L(0, \infty)$  and  $q < 0$ . Then  $L_2y$  is in the limit-point case if either  $L_2y$  or  $Mz$  is nonoscillatory.

The search for conditions on the coefficients of  $L_2y$  in order to place the operator in the limit-circle case appears to be less extensive. It may be that the limit-point case is more interesting since this case

makes the problem  $L_2 y = \lambda y$  self-adjoint. This is easily seen from the results of Theorem 4.4. For if  $L_2 y$  is in the limit-point case, then for all functions  $u$  and  $v$  in the domain of  $L_2$ , Theorem 4.4 implies

$$[u v](b) \rightarrow 0 \text{ as } b \rightarrow \infty.$$

Thus, for  $u$  and  $v$  in the set  $E$  of Theorem 4.4, an application of Theorem 1.3 yields

$$\begin{aligned} (L_2 u, v) - (u, L_2 v) &= \lim_{b \rightarrow \infty} \int_0^b \{(L_2 u) \bar{v} - u (\overline{L_2 v})\} dx \\ &= \lim_{b \rightarrow \infty} \{[u v](b) - [u v](0)\} \\ &= -[u v](0). \end{aligned} \tag{5.13}$$

As in the proof of Lemma 4.1,  $[u v](0) = 0$  and (5.13) implies  $(L_2 u, v) = (u, L_2 v)$ . Therefore,  $L_2 y$  with the boundary conditions of the set  $E$  of Theorem 4.4 is self-adjoint. If the limit-circle case holds, then  $L_2 y$  restricted to the set  $D(\infty)$  of Definition 4.1 is self-adjoint.

The following two theorems give conditions on the coefficients of  $L_2 y$  in order that the limit-circle case holds. The first theorem is more useful for constructing a class of examples. The second theorem is a generalization of a special case of the first result to the more general expression (5.4). The first theorem is due to Eastham [20]. It is noted that the techniques used to determine the limit-point case are less successful in determining the limit-circle case since most of the limit-point results are established by showing that at least one solution of  $L_2 y = 0$  is not in  $L^2(0, \infty)$ . It is a more difficult problem to show all solutions of  $L_2 y = 0$  are in  $L^2(0, \infty)$  if the solutions or asymptotic estimates of the solutions are not known. A proof of the

first theorem may be found in [20] and is an application of Theorem 4.4.

**THEOREM 5.9:** Let  $L_2 y$  be given by (5.4) and assume  $p(x) = 1$  for all  $x \geq 0$ . Let  $P$ ,  $Y$ , and  $h$  be real-valued functions defined on  $b \leq x < \infty$ ,  $b \geq 0$ , such that

- (i)  $P(x) > 0$  for  $x \geq b$  and  $P$  is in  $L^2(0, \infty)$ ,
- (ii)  $P'$  and  $Y$  are locally absolutely continuous on  $b \leq x < \infty$ ,
- (iii)  $Y(x) = o(1)$  as  $x \rightarrow \infty$ , and
- (iv)  $h$  and  $Y'/P$  are in  $L^2(b, \infty)$ .

Let  $q$  be defined by

$$q = h/P + P''/P - (1 + Y)/P^4,$$

for  $x \geq b$  and defined to be any  $L^2(0, b)$  function on  $0 \leq x < b$ . Then  $L_2 y$  is in the limit-circle case.

The following corollary of Theorem 5.9 is simple, but will be used in a later theorem.

**COROLLARY 5.3:** Let  $q_1(x)$  be a negative, decreasing function such that  $(-q_1)^{-1/4}$  is in  $L^2(0, \infty)$  and  $q_1$  has continuous derivatives of all orders. Then for  $q_2$  defined by

$$q_2 = q_1 + |q_1|^{1/4}(|q_1|^{-1/4})'',$$

the differential expression  $M_y$  defined by

$$M_y = -y'' + q_2 y$$

is in the limit-circle case.

**PROOF:** The corollary follows immediately by taking  $P = |q_1|^{-1/4}$ ,  $h = 0$ , and  $Y = 0$  in Theorem 5.9.

Note that there have been a number of theorems in which  $q_1$  can satisfy the hypothesis of Corollary 5.3 and  $Ly = -y'' + q_1 y$  may be in the limit-point case and (5.14) may be in the limit-circle case. Such a situation will be examined in a later theorem. The following theorem is due to Everitt [38] and is shown for the general expression (5.4). The corresponding result for  $p(x) = 1$  will follow from Theorem 5.9.

**THEOREM 5.10:** Let  $L_2 y$  be given by (5.4) and assume  $p'$  is continuous. If

- (i)  $p'$  and  $q'$  are locally absolutely continuous on  $0 \leq x < \infty$ ,
- (ii)  $p''$  and  $q''$  are in  $L^2(0, \infty)$  locally,
- (iii)  $q(x) < 0$  for  $x \geq 0$ ,
- (iv)  $(-pq)^{-1/4}$  is in  $L^2(0, \infty)$ , and
- (v)  $[p(pq)'(-pq)^{-5/4}]'$  is in  $L^2(0, \infty)$ ,

then  $L_2 y$  is in the limit-circle case.

**PROOF:** The proof will use Theorem 4.4. Define  $f$  by

$$f(x) = (-pq)^{-1/4} \exp\left\{i \int_0^x (-q/p)^{1/2} dt\right\}.$$

It is readily verified that the hypotheses of the theorem imply  $f$  satisfies the hypotheses of the second-order version of Theorem 4.4. A simple calculation will then show that  $[f f'](b) = -2i$ , and thus, by Theorem 4.4,  $L_2 y$  is in the limit-circle case. This completes the proof.

It is noted that in the case  $p(x) = 1$ , by letting  $P = (-q)^{-1/4}$ ,  $Y = 0$ , and  $h = -P''$ , Theorem 5.10 follows from Theorem 5.9.

The following result is due to Patula and Wong [68] and relates the limit-point case of a differential expression to a known differential expression. This result will also be used later.

THEOREM 5.11: Let  $L_2y$  be given by (5.4) and let  $My$  be given by

$$My = - (py')' + q_1y.$$

Assume  $L_2y$  is in the limit-point case. If all solutions of  $L_2y = 0$  are bounded and  $|q - q_1|$  is bounded, then  $My$  is in the limit-point case.

The next theorem is due to Eastham and Thompson [22]. The result of this theorem is quite remarkable and indicates the difficulty of the problem of determining necessary and sufficient conditions for the limit-point or limit-circle case to occur. The problem is shown to be quite delicate and it is possibly true that for this reason, no necessary and sufficient conditions on the coefficients of  $L_2y$  have yet been found that place the operator  $L_2y$  in the limit-point or limit-circle case. The proof given is a special case of the results of [22].

THEOREM 5.12: Given  $\epsilon > 0$ , there exist functions  $q_1$  and  $q_2$  that agree except on a sequence of intervals of total length of at most  $\epsilon$ , and such that for  $Ly$  and  $My$  defined by

$$Ly = - y'' + q_1y, \quad My = - y'' + q_2y, \quad (5.15)$$

$Ly$  is in the limit-point case and  $My$  is in the limit-circle case.

Furthermore,  $q_1$  and  $q_2$  can be taken to have continuous derivatives of all orders and  $q_1$  can be taken to be monotone.

PROOF: For each  $n = 1, 2, \dots$ , define

$$b_n = 2\pi/n, \quad s_n = \sum_{k=1}^n b_k.$$

Then let  $q$  be the step function defined by



$$q(x) = -n^2, \quad s_{n-1} \leq x < s_n,$$

where  $s_0 = 0$ . It will first be established that the differential expression  $L_2 y = -y'' + qy$  is in the limit-point case and that all solutions of  $L_2 y = 0$  are bounded. Let  $Y$  and  $Z$  be the functions given by

$$Y(x) = \cos(n[x - s_{n-1}]), \quad s_{n-1} \leq x < s_n,$$

$$Z(x) = \sin(n[x - s_{n-1}]), \quad s_{n-1} \leq x < s_n.$$

It is easily verified that  $Y$  and  $Z$  satisfy  $L_2 y = 0$  and are bounded. Also, since  $n(s_n - s_{n-1}) = nb_n = 2\pi$ , it is easily verified by elementary integration that

$$\int_0^\infty Y^2(x) dx = 2^{-1} \sum_{k=1}^\infty b_k = \infty,$$

implying  $Y$  is not in  $L^2(0, \infty)$ . Therefore,  $L_2 y$  is in the limit-point case.

Let  $q_1$  be any  $C^\infty(0, \infty)$  function satisfying the conditions

- (i)  $q_1$  is nondecreasing,
- (ii)  $q_1(x) \leq q(x)$  for each  $x \geq 0$ ,
- (iii)  $|q_1 - q|$  is bounded on  $0 \leq x < \infty$ ,
- (iv)  $q_1(x) = q(x)$  except in an  $\varepsilon 2^{-n-1}$  neighborhood of each  $s_n$ .

Then, since  $L_2 y$  is in the limit-point case and all solutions of  $L_2 y = 0$  are bounded, condition (iii) and Theorem 5.11 imply  $L y$  given in (5.15) is in the limit-point case.

Define  $q_2$  by

$$q_2(x) = q_1(x) + |q_1(x)|^{-1/4} \{ |q_1(x)|^{-1/4} \}''.$$

Recall that by the definition of  $q$  and condition (iv),  $q_1$  is constant

outside  $\varepsilon 2^{-n-1}$  neighborhoods of  $s_n$ . Hence, outside these neighborhoods,  $\{|q_1|^{-1/4}\}'' = 0$ , and thus  $q_2(x) = q_1(x)$  for values outside these neighborhoods. That is,  $q_1$  and  $q_2$  agree except on a set of measure less than  $\varepsilon$ . Since  $q_1$  is nonincreasing and negative,

$$|q_1(x)|^{-1/2} \leq |q_1(0)|^{-1/2}$$

For each  $n$ , let  $I_n$  denote the interval  $s_n - \varepsilon 2^{-n-1} \leq x \leq s_n + \varepsilon 2^{-n-1}$ . Then since  $q_1 = q$  on  $C(I_n)$ ,

$$\begin{aligned} \int_0^\infty |q_1(x)|^{-1/2} dx &\leq \int_{\cup I_n} |q_1(0)|^{-1/2} dx + \int_{C(\cup I_n)} |q_1(x)|^{-1/2} dx \\ &\leq \varepsilon |q_1(0)|^{-1/2} + \sum_{n=1}^\infty \int_{s_{n-1}}^{s_n} |q(x)|^{-1/2} dx \\ &= O(1) + \sum_{n=1}^\infty (s_n - s_{n-1})/n \\ &= O(1) + \sum_{n=1}^\infty b_n/n \\ &= O(1) + \sum_{n=1}^\infty 2\pi/n^2. \end{aligned}$$

Thus,  $|q_1(x)|^{-1/4}$  is in  $L^2(0, \infty)$ . Therefore, by Corollary 5.3, My given in (5.15) is in the limit-circle case. This completes the proof of the theorem.

Attention will now be centered on the fourth-order expression  $L_4 y$  given in (5.5). Fewer results have been established for this expression. One of the difficulties is that there is a case "between" the limit-point and limit-circle cases, namely the limit-3 case. The principal methods used to establish results in the fourth-order problem (and for higher order problems) are the use of Theorem 4.4 and the asymptotic method, although Hinton has published a fourth-order result using neither of these methods. The first theorem is an extension of

Corollary 5.2 in that it is assumed that the coefficient  $q$  of  $y$  is bounded. This result is due to Everitt [36] and a proof may be found in Theorem 1 of his paper.

**THEOREM 5.13:** Let  $L_{\mu}y$  be given by (5.5) and assume  $r(x) = 1$  for all  $x \geq 0$ . Suppose the coefficients  $p$  and  $q$  satisfy the conditions

- (i)  $q$  is locally integrable on  $0 \leq x < \infty$ ,
- (ii)  $p$  is locally absolutely continuous on  $0 \leq x < \infty$ ,
- (iii)  $p(x) \geq 0$  for  $x \geq 0$ ,
- (iv)  $q$  is bounded below, and
- (v) either  $0 \leq p(x) \leq Kx^2$  or  $0 \leq p(x) \leq Kx^2 |q(x)|^{1/2}$  for some  $K > 0$  and for all  $x \geq 0$ .

Then  $L_{\mu}y$  is in the limit-point case.

The following theorem, also due to Everitt [37], is similar to Theorem 5.13 in that the growth of the coefficients  $p$  and  $q$  is restricted. A proof of this result may be found in that paper.

**THEOREM 5.14:** Let  $L_{\mu}y$  be given by (5.5) and assume  $r(x) = 1$  for all  $x$ . Let  $E$  be as defined in Theorem 4.4 where the form  $[u \ v]$  is defined with respect to the differential expression  $L_{\mu}y$ . Let  $k$ ,  $l$ , and  $m$  be nonnegative constants and let the coefficients  $p$  and  $q$  of  $L_{\mu}y$  satisfy the conditions

- (i)  $q$  is locally integrable on  $0 \leq x < \infty$ ,
- (ii)  $p$  is locally absolutely continuous on  $0 \leq x < \infty$ ,
- (iii)  $q(x) \geq -kx^2$  almost everywhere for  $x \geq 0$ , and
- (iv)  $-lx^{2/3} \leq p(x) \leq mx^{10/3}$  for all  $x \geq 0$ .

Then  $L_{\mu}y$  is in the limit-point case.

The method of proof of the following theorem by Hinton [52] is unusual for a fourth-order problem in that it does not involve either Theorem 4.4 or the asymptotic method. Moreover, the result is easily applied. The statement given is for a special case of Hinton's theorem and the proof given in his paper is easily followed.

THEOREM 5.15: Let  $L_4 y$  be given by (5.5) and assume further that  $p(x) \geq 0$ ,  $q(x) \geq 1$ ,  $p(x) = O(x^2)$ , and  $r'(x) = O(x^3)$  as  $x \rightarrow \infty$ . Then  $L_4 y$  is in the limit-point case.

The asymptotic method will now be considered. Since the mid 1940's, the main work on the deficiency index problem has been done in England, Russia, and the United States. In England and in the United States, the specific problem of the deficiency index was studied only for second-order operators until the late 1960's. At that time the fourth-order problem was considered, primarily by Everitt, Hinton, Eastham, Devinatz, Walker, and Wood. In Russia, beginnings were made on the establishment of a theory for higher order operators. The Russian school used the asymptotic method in the early 1950's to obtain deficiency index theorems for higher order operators. An excellent account of some of these methods and results appears in Naimark's book [65].

In the United States, Everitt, Hinton, and Eastham generally used methods other than the asymptotic method while Devinatz, Walker, and Wood employed asymptotic methods. An example of the asymptotic method will be considered. This example is due to Walker [76, 77]. To attempt to survey all the results using this method would be too lengthy since the proofs of these results tend to be quite complicated and long. For the fourth-order problem, the most notable results are due to

Devinatz [13, 14, 15], Walker [76, 77, 78], and Wood [86]. The results presented by Naimark [65] for the  $2n$ -th order operator can also certainly be restricted to the fourth-order operator  $L_4 y$ .

The asymptotic method in the deficiency index problem is generally based on an asymptotic theorem of Levinson [7] or [8:p. 92] and certain extensions of this theorem due to Devinatz [11] and Fedorjuk [45]. Instead of trying to apply Levinson's theorem directly to the problem, the procedure is to make certain transformations on the independent and dependent variables in order that Levinson's theorem may be applied. This procedure will be generally described for the  $2n$ -th order problem. The differential equation to be considered is put into the form

$$U'(x) = A(x)U(x), \quad (5.16)$$

where the matrix  $A$  is given by (1.5). For convenience, assume the problem is defined on the interval  $0 \leq x < \infty$ . Let  $Q_0$  be a nonnegative measurable function such that  $1/Q_0$  is locally integrable on  $0 \leq x < \infty$ , but not integrable on the entire interval  $0 \leq x < \infty$ . Let

$$s(x) = \int_0^x (1/Q_0(t)) dt.$$

The function  $s$  is monotone increasing, locally absolutely continuous and has a monotone increasing inverse which may be denoted  $x = x(s)$ . By setting  $V(s) = U(x(s))$ , (5.16) yields

$$V'(s) = Q_0(x(s))A(x(s))V(s), \quad (5.17)$$

where the prime in each case will denote differentiation with respect to the indicated independent variable. Let  $Q_1, \dots, Q_n$  be positive functions on the interval  $0 \leq x < \infty$  which are all locally absolutely



where  $B$  is a constant matrix with simple eigenvalues,  $V(s) = o(1)$  as  $s \rightarrow \infty$ , and  $V'(s)$  and  $R(s)$  are integrable on some interval  $a \leq s < \infty$ ,  $a \geq 0$ . With these hypotheses and under other suitable conditions, the theorem of Levinson can be applied. This theorem states that there is a complete set  $\{w_k\}$  of solutions of (5.18) and an  $s_0 > 0$  such that

$$w_k(s) \exp \left\{ - \int_{s_0}^s \lambda_k \right\} \rightarrow e_k,$$

where  $e_k$  is a complete set of eigenvectors for  $B$ . Thus, an estimate on the growth of the solutions  $w_k$  can be given, and transforming back, estimates on the growth of a complete set of solutions of  $L_{2n}y = \lambda y$  can be made.

The asymptotic method is not always applicable. Generally, when the coefficients have "large" oscillations, it is not possible to transform the problem into one which is a small perturbation of a differential operator with constant coefficients. Even if such a transformation is possible, the constant matrix  $B$  of the decomposition (5.19) may have multiple eigenvalues. The problem of finding asymptotic estimates in the latter case is not easy, and only recently have some beginnings been made by Devinatz and Walker. It is a problem of the latter type that will be considered. The asymptotic theorem of Levinson's is Theorem 1, page 88, of [8]. It is this theorem that will be used. Before stating this theorem, a lemma and a definition will be stated.

**DEFINITION 5.1:** Let  $b$  be a real number and  $D$  a real-valued continuous function defined on  $b \leq x < \infty$ . Then  $D$  is said to satisfy condition (\*) if and only if either

(i)  $\int_b^x D(t) dt \rightarrow \infty$  as  $x \rightarrow \infty$  and there is a real number  $K$  such that if  $b \leq x_1 \leq x_2$ , then  $\int_{x_1}^{x_2} D(t) dt > K$ , or

(ii) there exists a real number  $K$  such that if  $b \leq x_1 \leq x_2$ , then  $\int_{x_1}^{x_2} D(t) dt < K$ .

Note that  $K$  need not be positive. Condition (\*) will be used in the theorems to follow and it will be useful to have some conditions that imply condition (\*). The proof of the following lemma is elementary.

**LEMMA 5.1:** Let  $b_0$  and  $b_1$  be real numbers with  $b_0 \leq b_1$  and let  $D$  be a continuous function defined on  $b_0 \leq x \leq \infty$ . Then, each of the following implies  $D$  satisfies condition (\*).

- (i) The restriction of  $D$  to  $x \geq b_1$  satisfies condition (\*).
- (ii)  $D$  is nonnegative, negative, nonpositive, or positive for all  $x \geq b_1$ .
- (iii) There exists a monotone function  $m$  and a bounded function  $w$  such that for  $b_1 \leq x_1 \leq x_2$ ,

$$\int_{x_1}^{x_2} D(t) dt = [m(t) + w(t)]_{x_1}^{x_2}.$$

- (iv)  $D = D_1 + D_2$  for  $x \geq b_1$  where  $D_1$  satisfies condition (\*) and  $D_2$  is integrable on  $b_1 \leq x < \infty$ .

Following is the variation of Levinson's theorem that will be used. In this theorem and in the next, capital letters will denote matrices or vectors and lower case letters will denote real or complex valued functions. In Theorem 1, page 88, of [8], a condition that  $\operatorname{Re}(\lambda_k - \lambda_j)$  does not change sign is made. In the following theorem, this condition



is replaced by a condition that  $\operatorname{Re}(\lambda_k - \lambda_j)$  satisfies condition (\*). It is easily seen that the theorem remains valid under this weaker condition. Theorem 8.1, Chapter 3, of [7] uses this weaker condition.

**THEOREM 5.16:** Let  $T$  be the diagonal matrix

$$T(x) = \operatorname{diag} [\lambda_1(x), \dots, \lambda_n(x)]$$

and let  $F$  be a continuous matrix such that  $\|F(x)\|$  is integrable on the interval  $b \leq x < \infty$ . For a fixed  $k$ ,  $1 \leq k \leq n$ , define

$$D_{kj}(x) = \operatorname{Re}(\lambda_j(x) - \lambda_k(x)), \quad 1 \leq j \leq n.$$

If all the functions  $D_{kj}$  satisfy condition (\*), then the differential equation

$$Y' = [T(x) + F(x)] Y$$

has a solution  $Y_k(x)$  such that as  $x \rightarrow \infty$ ,

$$Y_k(x) = E_k \exp \left\{ \int_b^x \lambda_k(t) dt \right\} + o(1),$$

where  $E_k$  is the elementary vector with zeros in each position except the  $k$ -th position which is one.

Before presenting the theorem, a preliminary lemma will be needed. The lemma follows immediately by performing the indicated differentiation.

**LEMMA 5.2:** Let each of  $S$  and  $T$  be a nondegenerate connected subset of the real line. Suppose there is a continuously differentiable homeomorphism  $h: S \rightarrow T$  such that  $h'$  does not vanish on  $T$ . Let  $g$  be the

function inverse of  $h$ . For  $M_n$  the set of  $n \times n$  complex matrices, let  $A: T \rightarrow M_n$  be a continuous function. Let  $Q: T \rightarrow M_n$  be a continuously differentiable function such that  $Q^{-1}(t)$  exists for each  $t$  in  $T$ . Then, if  $Y_0: T \rightarrow M_n$  is a fundamental matrix for  $Y' = AY$ , then  $Z_0: S \rightarrow M_n$ , defined by  $Z_0 = Q(g)Y_0(g)$ , is a fundamental matrix for

$$Z' = (1/h'(g)) [Q(g)A(g)Q^{-1}(g) + Q'(g)Q^{-1}(g)] Z.$$

The following theorem employing the asymptotic method is due to Walker [76]. Although he established the theorem for more general coefficients  $r$ ,  $p$ , and  $q$  on  $L_4y$ , the theorem will be established for the particular case

$$L_4y = (x^a y'')'' - (-1)^j (x^b y')' \pm x^c y, \quad x \geq 1 \quad (5.20)$$

since the general theorem is quite complicated and is less suited to the deficiency index problem. Particular differential expressions of the form (5.20) have been studied in recent years since they hope to give an indication of how "near" the sufficient conditions of such results as the preceding fourth-order results are to being necessary. This particular result will show the results of Theorem 5.13 are not the best possible for  $L_4y$  given by (5.20).

**THEOREM 5.17:** Let  $L_4y$  be given by (5.20) with  $a = 0$ ,  $c = 0$ , and  $b = 4$ . Then the differential expression  $L_4y$  is in the limit-2 (limit-point) case if  $j = 2$ .

**PROOF:** Let the function  $h: [1, \infty) \rightarrow [0, \infty)$  be given by the equation  $h(x) = (1/3)(x^3 - 1)$  and let  $g: [0, \infty) \rightarrow [1, \infty)$  be given by the equation  $g(s) = (3s + 1)^{1/3}$ . Note that  $g$  is the function inverse of  $h$ .

For  $s \geq 0$ , define the functions  $\alpha$  and  $\beta$  by

$$\alpha(s) = 4(3s + 1)^{-1}, \quad \beta(s) = (3s + 1)^{-8/3}.$$

Then it is easily seen that  $\alpha'$ ,  $\beta'$ , and  $\alpha^2$  are in  $L(0, \infty)$ . Let  $v > 1$  be such that  $s^v \beta$  is in  $L(0, \infty)$ . Define  $E: [0, \infty) \rightarrow M_4$  by

$$E(s) = \text{diag} [e^{-s}, e^s, s^{-v}(3s + 1), (3s + 1)^{-1}] \quad (5.21)$$

and define  $G: [1, \infty) \rightarrow M_4$  by

$$G(x) = \text{diag} [x^3, x, x^{-1}, x^{-3}h^v(x)].$$

Let  $A$  be the matrix

$$A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -x^4 & 0 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}.$$

By Theorem 1.2, the system

$$Y' = AY \quad (5.22)$$

is equivalent to the differential equation  $L_h y = iy$ . It will be shown that there exists a fundamental matrix  $Y_0$  for (5.22) such that as  $x \rightarrow \infty$ ,

$$G(x)Y_0(x)E(h(x)) = K + o(1) \quad (5.23)$$

where  $K$  is the matrix

$$K = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Let  $Y_1$  be a fundamental matrix for (5.22) and define  $Q: [1, \infty) \rightarrow M_4$  by

$$Q(x) = \text{diag} [x^3, x, x^{-1}, x^{-3}]$$

and let  $Z_1$  be given by

$$Z_1(s) = Q(g(s))Y_1(g(s)). \quad (5.24)$$

By Lemma 5.2,  $Z_1$  is a fundamental matrix for the system

$$Z' = (1/h'(g)) [Q(g)A(g)Q^{-1}(g) + Q'(g)Q^{-1}(g)] Z. \quad (5.25)$$

Note that (5.25) may be expressed as

$$Z' = [A_0 + V] Z \quad (5.26)$$

where  $A_0$  and  $V$  are given by

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 3\alpha/4 & 0 & 0 & 0 \\ 0 & \alpha/4 & 0 & 0 \\ 0 & 0 & -\alpha/4 & 0 \\ 0 & 0 & 0 & -3\alpha/4 \end{bmatrix}.$$

Let  $J$  be the matrix

$$J = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Let

$$W_1(s) = J^{-1}Z_1(s) \quad (5.27)$$

and note that by (5.26),

$$W_1' = (J^{-1} [A_0 + V] J) W_1. \quad (5.28)$$

Also, note that  $W_1$  is nonsingular. By a calculation,

$$J^{-1}A_0J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J^{-1}VJ = \begin{bmatrix} 0 & 0 & -\alpha/2 & 0 \\ 0 & 0 & \alpha/2 & 0 \\ 0 & 0 & -3\alpha/4 & 0 \\ \alpha & \alpha & 0 & 3\alpha/4 \end{bmatrix}.$$

For  $s \geq 1$ , define  $p$  by

$$P(s) = \text{diag} [1, 1, s^V, 1]$$

and let  $X_1(s)$  be given for  $s \geq 1$  by

$$X_1(s) = P(s)W_1(s). \quad (5.29)$$

Then it follows from (5.28) and (5.29) that

$$X_1' = [P'P^{-1} + PJ^{-1}A_0P^{-1} + PJ^{-1}VJP^{-1}] X_1. \quad (5.30)$$

Also, note that  $X_1$  is nonsingular. Expression (5.30) may be expressed

as

$$X_1' = [B + C] X_1, \quad (5.31)$$

where B and C are given by

$$B = \begin{bmatrix} 1 & -\alpha/4 & 0 & 0 \\ -\alpha/4 & -1 & 0 & 0 \\ 0 & 0 & vs^{-1} & -3\alpha/4 \\ \alpha & \alpha & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & -\alpha s^{-v}/2 & 0 \\ 0 & 0 & \alpha s^{-v}/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & s^{-v} & 0 \end{bmatrix}.$$

From the choice of  $v$  and the definition of  $\alpha$ , it is clear that  $C$  is in  $L(1, \infty)$ . Let  $R$  be the function defined as  $R: \mathcal{C} \times [1, \infty) \rightarrow \mathcal{C}$ , where  $R(z, s) = \det(B - zI_4)$ . Then a computation shows

$$R(z, s) = (3\alpha/4 - z)(-3\alpha/4 + vs^{-1} - z)(z^2 - 1 - \alpha^2/16).$$

Let  $\lambda_1(s)$  and  $\lambda_2(s)$  be the two continuous functions satisfying  $\lambda_k^2(s) - 1 - \alpha^2(s)/16 = 0$ ,  $k = 1$  and  $2$ . Then  $\lambda_1(s) = 1 + o(1)$  and  $\lambda_2(s) = -1 + o(1)$  as  $s \rightarrow \infty$  since  $\alpha(s) = o(1)$ . Also, define  $\lambda_3$  and  $\lambda_4$  by

$$\lambda_3(s) = -3\alpha(s)/4 + vs^{-1}, \quad \lambda_4(s) = 3\alpha(s)/4, \quad (5.32)$$

and note that  $R(\lambda_k(s), s) = 0$  for  $1 \leq k \leq 4$ . Let  $S$  be the matrix

$$S = \begin{bmatrix} 1+\lambda_1 & -\alpha/4 & 0 & 0 \\ \alpha/4 & -1-\lambda_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha(1+\lambda_4-\alpha/4) & \alpha(\lambda_4-1-\alpha/4) & 0 & 1+\lambda_4^2-\alpha^2/16 \end{bmatrix}.$$

Then by using the definitions of  $\lambda_1$  and  $\lambda_2$ , it is easily calculated that

$$SB = \text{diag} [\lambda_1, \lambda_2, \lambda_3, \lambda_4] S.$$

From the definitions of  $\alpha$ ,  $\lambda_1$ , and  $\lambda_2$ , it follows that  $\|S'\|$  is integrable on  $1 \leq s < \infty$ . Also, note that as  $s \rightarrow \infty$ ,  $\alpha = o(1)$ , thus

$$S(s) = \text{diag} [2, -2, 1, 1] + o(1). \quad (5.33)$$

Therefore,  $S$  is nonsingular for sufficiently large  $s$ . Applying Lemma 5.2 again and using (5.31) while letting

$$U_1(s) = S(s)X_1(s), \quad (5.34)$$

for  $s \geq s_1 \geq s_0$ , it follows that  $U_1$  is a fundamental matrix for the system

$$U' = [SBS^{-1} + SCS^{-1} + S'S^{-1}] U, \quad (5.35)$$

where, for this application of the lemma,  $h(s) = s$ . By consideration of (5.33), both  $S$  and  $S^{-1}$  are bounded for  $s \geq s_1$ . Thus, since  $\|C\|$  is integrable on  $s \geq s_1$  and  $\|S'\|$  is integrable on the same interval, it follows that  $\|SCS^{-1} + S'S^{-1}\|$  is integrable on  $s_1 \leq s < \infty$ . Therefore, the system (5.35) may be expressed as

$$U' = [T + F] U, \quad (5.36)$$

where

$$T = \text{diag} [1, -1, \lambda_3, \lambda_4],$$

$$F = \text{diag} [\lambda_1 - 1, \lambda_2 + 1, 0, 0] + SCS^{-1} + S'S^{-1}.$$

It is easily seen that  $\|F\|$  is in  $L(s_1, \infty)$ . Also, it is easily seen by using Lemma 5.1, for  $D_{km}$  the real part of the  $k$ -th diagonal element of  $T$  minus the  $m$ -th,  $D_{km}$  satisfies condition (\*) for all  $k$  and all  $m$ . Therefore, by Theorem 5.16, there is a fundamental matrix  $U_0$  for the

system (5.36) such that as  $s \rightarrow \infty$ ,

$$U_0(s) \exp \left\{ - \int_{b_1}^s \text{diag} [1, -1, \lambda_3(t), \lambda_4(t)] dt \right\} = I_4 + o(1). \quad (5.37)$$

By evaluating the integral using (5.32), the exponential factor in (5.37) is given by  $DE(s)$  where  $D$  is the constant diagonal matrix obtained by evaluating the integral at  $s_1$ . Therefore, by (5.37),

$$U_0(s)DE(s) = I_4 + o(1).$$

The reverse transformations will be made.  $D$  is nonsingular and so each of  $U_0D$  and  $U_1$  are fundamental matrices for the system (5.36). Let  $H$  be a constant nonsingular matrix such that  $U_0D = U_1H$ . Then by (5.34), (5.29), (5.27), and (5.24), for  $s = h(x)$ ,

$$U_1 = SX_1 = SPW_1 = SPJ^{-1}Z_1 = SPJ^{-1}QY_1. \quad (5.38)$$

Let  $Y_2 = Y_1H$  on the interval  $[g(s_1), \infty)$  and extend this to a fundamental matrix of (5.22) on all of  $[1, \infty)$ . Then from (5.38),

$$U_0DE = U_1HE + SPJ^{-1}QY_1HE = SPJ^{-1}QY_2E. \quad (5.39)$$

By (5.39), as  $s \rightarrow \infty$  ( $h(x) \rightarrow \infty$ ),

$$SPJ^{-1}QY_2E = I_4 + o(1). \quad (5.40)$$

Since  $J^{-1}$  is constant and

$$S^{-1}(x) \rightarrow \text{diag} [1/2, 1/2, 1, 1],$$

(5.40) implies

$$PJ^{-1}QY_2E = S^{-1}(\infty) + o(1),$$



that is,

$$JPJ^{-1}QY_2E = JS^{-1}(\infty) + o(1).$$

It is easily calculated that

$$JPJ^{-1} = K_1 \text{diag} [1, 1, 1, h^v],$$

where

$$K_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & h^{-v}-1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore,

$$K_1 \text{diag} [1, 1, 1, h^v] QY_2E = JS^{-1}(\infty) + o(1),$$

that is,

$$\text{diag} [1, 1, 1, h^v] QY_2E = K_1^{-1}JS^{-1}(\infty) + o(1).$$

Now, let

$$Y_0(x) = Y_2(x)S^{-1}(\infty)$$

and note that  $\text{diag} [1, 1, 1, h^v] Q = G$ . Since diagonal matrices commute among themselves,

$$G(x)Y_0(x)E(h(x)) = K_1^{-1}(\infty)J + o(1) = K + o(1).$$

Therefore, (5.23) is established. The conclusion of the theorem may now be shown. The first row of the product  $GY_0E$  will be compared with the first row of  $K$ . From (5.21), using  $g(h(x)) = x$  and  $h(g(s)) = s$ ,

$$E(x) = \text{diag} [e^{-h(x)}, e^{h(x)}, h^{-v}(x), x^{-3}].$$

Then by comparing the first and fourth columns, there are solutions  $y_1$  and  $y_2$  such that

$$y_1 = x^{-3} e^{h(x)} (1 + o(1)), \quad y_2 = 1 + o(1).$$

Since  $h(x)$  tends to infinity with  $x$ ,  $y_1$  tends to infinity and  $y_1$  is not in  $L^2(1, \infty)$ . Also, it is clear that no linear combination of  $y_1$  and  $y_2$  can be in  $L^2(1, \infty)$ . Therefore, the limit-point case holds and the proof of Theorem 5.17 is complete.

Theorem 5.17 implies the differential expression

$$L_4 y = y^{iv} - (x^4 y')' + y$$

is in the limit-point case and thus the condition

$$0 \leq p(x) \leq Kx^2 \text{ or } 0 \leq p(x) \leq Kx^2 |q(x)|^{1/2},$$

where  $p(x) = x^4$  and  $q(x) = 1$ , of Theorem 5.13 is not nearly a necessary condition for the limit-point case to hold.

This survey of the deficiency index problem will be concluded by some results that connect certain results for the second and fourth-order cases. This connection will be considered by examining the limit-p properties of the fourth-order operator obtained by "squaring" the second-order operator, that is, for sufficiently differentiable coefficients  $p$  and  $q$ ,

$$L^2 y = L(Ly),$$

where  $Ly$  is the operator defined by (5.4).

Let  $p$  and  $q$  be real-valued functions such that  $p(x) > 0$  for all  $x \geq 0$ ,  $q$  is locally integrable on  $0 \leq x < \infty$ , and  $p, p', p'', q$ , and  $q'$  are all locally absolutely continuous on  $0 \leq x < \infty$ . Then (5.41) can be put into the form

$$L^2 y = (p^2 y'')'' - (2pq - pp'' y')' + (q^2 - pq'' - p'q')y.$$

The first of these results is quite easily established and the results are due to Chaudhuri and Everitt [5].

**THEOREM 5.18:** The differential expression  $L^2 y$  is in the limit-4 case if and only if  $Ly$  is in the limit-circle case.

**PROOF:** Suppose  $Ly$  is in the limit-circle case. Let  $u(x, i)$  and  $v(x, i)$  be two linearly independent solutions of  $Ly = iy$ . Then, since  $Ly$  is in the limit-circle case, both  $u(x, i)$  and  $v(x, i)$  are in  $L^2(0, \infty)$ . Also, since the coefficients of  $Ly$  are real-valued,  $\bar{u}(x, i) = u(x, -i)$  and  $\bar{v}(x, i) = v(x, -i)$  where  $u(x, -i)$  and  $v(x, -i)$  are solutions of  $Ly = -iy$ . Clearly, both  $u(x, -i)$  and  $v(x, -i)$  are also in  $L^2(0, \infty)$ . Then,

$$L^2 u(x, i) = L(Lu(x, i)) = L(iu(x, i)) = -u(x, i),$$

$$L^2 u(x, -i) = L(Lu(x, -i)) = L(-iu(x, -i)) = -u(x, -i).$$

Similar relations hold for  $v$ . Thus, the four functions  $u(x, \pm i)$  and  $v(x, \pm i)$  are solutions of  $L^2 y = -y$ . It is easily shown that the four solutions are linearly independent. Since the limit-4 case is independent of the parameter  $\lambda$ ,  $L^2 y$  is in the limit-4 case.

The argument is reversible. If  $L^2 y$  is in the limit-4 case, then all solutions of  $L^2 y = -y$  are in  $L^2(0, \infty)$ . In particular, for  $u(x, i)$

and  $v(x, i)$  linearly independent solutions of  $Ly = iy$ ,  $u(x, \pm i)$  and  $v(x, \pm i)$  are four linearly independent solutions of  $L^2y = -y$  and are in  $L^2(0, \infty)$ . In particular, the solutions  $u(x, i)$  and  $v(x, i)$  of  $Ly = iy$  are in  $L^2(0, \infty)$ . Therefore,  $Ly$  is in the limit-circle case and the proof is complete.

Let  $[u \ v]$  denote the bilinear form associated with  $Ly$  and  $[u \ v]_2$  the bilinear form associated with  $L^2y$ . The expression  $Ly$  will be assumed to be in the limit-point case. The following theorem gives a characterization for  $L^2y$  to be in the limit-2 case. Since part of the proof is in terms of the theory of self-adjoint operators, some sets will need to be defined. Let  $\phi$  be the boundary value function defined for the second-order problem and  $\phi_1$  and  $\phi_2$  the boundary value functions for the fourth-order problem.

Let  $D(T)$  be the set of all functions  $f$  satisfying the following conditions.

- (i)  $f$  is in  $L^2(0, \infty)$ ,
- (ii)  $f'$  is locally absolutely continuous on  $0 \leq x < \infty$ ,
- (iii)  $Lf$  is in  $L^2(0, \infty)$ , and
- (iv)  $[f \ \phi](0) = 0$ .

Then define  $T$  by  $T(f) = Lf$  for  $f$  in  $D(T)$ .

Let  $D(T^2)$  be the set of all functions  $f$  satisfying the following conditions.

- (i)  $f$  is in  $D(T)$ , and
- (ii)  $Tf$  is in  $D(T)$ .

Then define  $T^2$  by  $T^2(f) = T(Tf)$  for  $f$  in  $D(T^2)$ .

Let  $D(S)$  be the set of all functions  $f$  satisfying the following conditions.

- (i)  $f$  is in  $L^2(0, \infty)$ ,
- (ii)  $f''$  is locally absolutely continuous on  $0 \leq x < \infty$ ,
- (iii)  $L^2 f$  is in  $L^2(0, \infty)$ , and
- (iv)  $[f \phi_1]_2(0) = 0, i = 1, 2.$

Then define  $S$  by

$$S(f) = L^2 f \quad (5.42)$$

for  $f$  in  $D(S)$ .

Assume  $T$  is self-adjoint and  $u$  and  $v$  are functions in  $D(T^2)$ . From the definition of  $D(T^2)$ , these functions are in  $D(T)$  and  $Lu$  and  $Lv$  are also in  $D(T)$ . Therefore,

$$(T^2 u, v) = (T(Tu), v) = (Tu, Tv) = (u, T^2 v),$$

implying  $T^2$  is self-adjoint. Also, if  $L_y$  is in the limit-point case, the second-order version of Theorem 4.4 implies  $T$  is self-adjoint. To see this, let  $u$  and  $v$  be in  $D(T)$ . Then applying Theorem 3.2 with  $f_1 = \bar{g}_1 = \phi$ ,  $f_2 = \bar{g}_2 = u$ , and  $f_3 = \bar{g}_3 = v$  yields the relation  $[u v](0) = 0$ . By Theorem 4.4 and Theorem 1.3,

$$(u, Lv) - (Lu, v) = [u v](\infty) - [u v](0) = 0,$$

implying  $T$  is self-adjoint. A similar result holds for the operator  $S$ , that is, if  $L^2 y$  is in the limit-point case, then  $S$  is self-adjoint.

It is readily calculated by Definition 1.5 that

$$[u v]_2(x) = [u Lv](x) + [Lu v](x) \quad (5.43)$$

when certain differentiability conditions are assumed on  $p$  and  $q$ . For example, involved in the second-order quasi-derivative is the

expression  $(py')'$ . This can be expanded to  $p'y' + py''$  and (5.43) can then be established with elementary calculations. The characterization that  $L^2y$  be in the limit-2 case can now be given. The proof of this theorem is given in Section 7 of [5] and is established by showing that  $D(T^2) = D(S)$  under the conditions of the theorem.

**THEOREM 5.19:** Let  $Ly$  be in the limit-point case and let  $S$  and  $D(S)$  be as defined in (5.42). Then  $L^2y$  is in the limit-2 case if and only if  $f$  in  $S$  implies  $Lf$  is in  $L^2(0, \infty)$ .

The following theorem gives a necessary and sufficient condition for  $L^2y$  to be in the limit-3 case in terms of the solutions of  $Ly = \lambda y$ . Its proof is given in Section 8 of [5].

**THEOREM 5.20:** Let  $Ly$  be in the limit-point case and let  $\phi(x, \lambda)$  be that solution of  $Ly = \lambda y$  such that for all  $\lambda$ ,

$$\phi(0, \lambda) = 0, \phi'(0, \lambda) = -1.$$

Then the differential expression  $L^2y$  is in the limit-3 case if and only if there is a value of  $\lambda$  with

$$-\pi/2 < \text{Arg } \lambda < \pi/2, \text{Arg } \lambda \neq 0$$

and a complex constant  $k$  such that

$$\phi(x, \lambda) + k\phi(x, -\lambda)$$

is in  $L^2(0, \infty)$ .

The following theorem is an application of the previous theorems on the square of  $Ly$ .

**THEOREM 5.21:** Let  $Ly = -y'' + qy$  and assume  $q'$  is locally absolutely continuous on  $0 \leq x < \infty$ . Then  $Ly$  is in the limit-point case and  $L^2y$  is in the limit-2 case if either

- (i)  $q(x) \geq k_1$  for all  $x \geq 0$  and  $q''(x) \leq kq^2(x)$  almost everywhere for  $x \geq 1$  where  $0 \leq k_1 < \infty$  and  $0 < k < 1$ , or
- (ii)  $-k_2x^{2/3} \leq q(x) \leq k_3x^{2/3}$  for all  $x \geq 0$  and  $q''(x) \leq k_4x^{4/3}$  for almost all  $x \geq 0$ , where  $0 \leq k_2, k_3, k_4 < \infty$ .

In Theorem 5.21, condition (i) and Theorem 5.7 imply  $Ly$  is in the limit-point case. Condition (ii) and Theorem 5.1 imply  $Ly$  is in the limit-point case. Condition (i) and Theorem 5.13 imply  $L_2y$  is in the limit-2 case and condition (ii) and Theorem 5.14 imply  $L^2y$  is in the limit-2 case.

Some examples illustrating all the possibilities of  $Ly$  and  $L^2y$  are given. If  $Ly$  is given by

$$Ly = -y'' + qy \quad (5.44)$$

with  $q$  given by any of

$$q(x) = 0, \quad q(x) = x^2, \quad q(x) = -(x+1)^{1/2},$$

then  $Ly$  is limit-point and  $L^2y$  is limit-2. In each of the three cases, (5.44) is limit-point by applying Theorem 5.1 and  $L^2y$  is limit-2 by Theorem 5.21.

For  $Ly$  given by (5.44) and  $q$  given by

$$q(x) = -e^x + 1/16,$$

$Ly$  is limit-circle and  $L^2y$  is limit-4. That  $Ly$  is limit-circle follows

from Corollary 5.3 by taking  $q_1(x) = -e^x$  in the definition of  $q_2$  and letting  $q = q_2$ .  $L^2y$  is limit-4 by Theorem 5.18.

Let  $Ly$  be given by (5.4) with

$$p(x) = (1/6)(x+1)^4, \quad q(x) = (x+1)^2.$$

Then  $q$  is clearly bounded below and thus by Corollary 5.2,  $Ly$  is in the limit-point case. By Theorem 5.18,  $L^2y$  is not in the limit-4 case.

Let  $\alpha = -(1/2)(7 - \sqrt{33})$  and let

$$f(x) = (x+1)^\alpha.$$

It is easily calculated that for the given value of  $\alpha$ ,  $L^2f = 0$ . Also,  $-\alpha < -1$  and thus  $f$  is in  $L^2(0, \infty)$ . It is also easily verified that

$$Lf = -(1/6)(\alpha^2 + 3\alpha - 6)(x+1)^{\alpha+2}.$$

Then, since  $\alpha + 2 > 0$ ,  $Lf$  is not in  $L^2(0, \infty)$ . Clearly  $f'''$  is locally absolutely continuous on  $0 \leq x < \infty$ . Redefine  $f$ , if necessary, on the interval  $0 \leq x \leq 1$  such that condition (iv) of the definition of  $D(S)$  holds. Then for this new function, say  $g$ ,  $g$  and  $L^2g$  are in  $L^2(0, \infty)$  and  $g$  satisfies condition (iv) of the definition of  $D(S)$ . Therefore,  $g$  is in  $D(S)$ . Now, if  $L^2y$  were in the limit-2 case, Theorem 5.19 would imply  $Lg$  is in  $L^2(0, \infty)$ . That is,  $Lf$  is in  $L^2(0, \infty)$ , a contradiction. Thus,  $L^2y$  is in the limit-3 case.



## CHAPTER VI

### SUMMARY

The primary purpose of this paper has been to trace the development of the deficiency index problem and to present a justification for its existence. Fourier series were generalized by Sturm and Liouville so as to cover the eigenvalues and eigenfunctions of the formally self-adjoint differential expression

$$L_{\lambda} y = - (py')' + (q - \lambda)y = 0 \quad (6.1)$$

subject to real linear boundary conditions at either end of a compact interval  $0 \leq x \leq b$ ,  $b > 0$ . The deficiency index problem had its beginnings in Hermann Weyl's [79] investigations of the generalizations to a singular interval of the Sturm-Liouville expansions associated with (6.1). As was demonstrated in Chapter II, the extension of the problem to the semi-infinite interval  $0 \leq x < \infty$  induced a classification of the differential equation (6.1) into one of two families. Membership in one of these families is determined by whether (6.1) is in the limit-point or limit-circle case, that is, whether or not all solutions of (6.1) lie in  $L^2(0, \infty)$ . These cases are determined geometrically by "contracting circles" in the complex plane and are independent of the complex parameter  $\lambda$ .

W. N. Everitt [30] extended the deficiency index problem to formally self-adjoint differential expressions of any even order. His

development was presented in Chapter III for these higher order problems by restricting attention to the fourth-order case. This restriction still contains all of the difficulties present in higher order problems. That is, generalization of the development to problems of order  $2n$ ,  $n > 2$ , is a matter of notation while the extension from second-order to fourth-order introduces some problems caused by the existence of cases "between" the limit-point and limit-circle cases. These difficulties may be seen by comparing the results of Chapter II with those of Chapter III. The results of Chapter II follow immediately from those of Chapter III, but the second-order case was presented separately since the ideas are more intuitive and geometric and the development is simpler.

Separate development of a second-order expansion theorem was omitted even though the restrictions from fourth-order expansions to those of the second order have the advantages of simplicity. However, the order of simplification from fourth-order expansion theorems to those of second order is not sufficient to justify a separate development. In other words, the establishment of a second-order expansion theorem is almost as complicated as the establishment of the fourth-order theorem.

The primary goal of this paper is contained in Chapter V. The intention has been to present in a unified manner various results on necessary and sufficient conditions for the various limit-p cases to occur and to examine various techniques employed to establish these results. The presentation of these results and techniques may serve to aid in the further investigation toward the ultimate goal of determining necessary as well as sufficient conditions for the determination of the deficiency index of a particular problem. This ultimate goal would

seem to be very difficult to obtain in light of Theorem 5.13 due to Eastham and Thompson [22]. Recall that this theorem states that for the differential expression  $-y'' + qy$ , the coefficient  $q$  may be re-defined on a set of arbitrarily small positive measure and change the deficiency index.

As discussed in Chapter V, the primary method for considering differential expressions of order higher than two is the asymptotic method. As seen in the presentation of the example of this method, this technique can be quite difficult to apply and in fact may not always be applicable. Thus, it would seem that one of the primary problems in the area of the investigation of higher order problems would be to establish a technique, other than the asymptotic method, for classifying a differential expression that is not in the limit-point case, that is, the deficiency index is not half the order of the expression. Recall that Theorem 4.4 only determines whether the expression is or is not in the limit-point case.

Investigation of the deficiency index problem has not been restricted to differential expressions with real-valued coefficients. Everitt [27, 28, 32, 33, 35] has considered the problem for differential expressions having complex-valued coefficients. This problem is more difficult since the conjugation operator does not necessarily commute with the operator determined by the differential expression as has been the case of real-valued coefficients. It would be of interest to determine those results that carry over from the real-valued coefficient problem. The complex-valued coefficient problem was not considered in the interest of compactness of presentation.

In recent years, the problem of classifying differential

expressions into various limit- $p$  cases has been extended to concepts of whether the differential expression (6.1) is in the strong (weak) limit point case or whether it is separated in  $L^2(0, \infty)$  space [2, 40, 41, 42, 43, 44]. Recall that Theorem 4.4 implies (6.1) is in the limit-point case if and only if the expression  $[u v](b)$  tends to zero as  $b$  tends to infinity for certain functions  $u$  and  $v$ . To illustrate the terms used above, definitions will be formulated for the second-order differential expression

$$Ly = -y'' + qy \quad (6.2)$$

where the coefficient  $q$  belongs to the class  $L^p(0, \infty)$  for some  $p$ ,  $1 \leq p < \infty$ .

DEFINITION 6.1: Let  $f$  be a function in  $L^2(0, \infty)$  and let  $Ly$  be given by (6.2). Then  $f$  belongs to the class  $D(q)$  if and only if

- (i)  $f$  is in  $L^2(0, \infty)$ ,
- (ii)  $f$  and  $f'$  are locally absolutely continuous on  $0 \leq x < \infty$ ,
- (iii)  $Lf$  is in  $L^2(0, \infty)$ .

Recall that by Definition 4.4 and the definition of the form  $[u v]$ , (6.2) is in the limit-point case if and only if

$$\lim_{b \rightarrow \infty} [f(b)\bar{g}'(b) - f'(b)\bar{g}(b)] = 0. \quad (6.3)$$

DEFINITION 6.2: The differential expression  $Ly$  of (6.2) is said to be strong limit-point if and only if for each pair of functions  $f$  and  $g$  in the class  $D(q)$ ,

$$\lim_{b \rightarrow \infty} f(b)\bar{g}'(b) = 0. \quad (6.4)$$

Clearly, if  $L_y$  is strong limit-point, then  $L_y$  is limit-point.  $L_y$  is called weak limit-point if it is limit-point without being strong limit-point. The concept of a differential expression to be separated in  $L^2(0, \infty)$  is basically a modification of condition (iii) of Definition 6.1.

DEFINITION 6.3: Let  $L_y$  be given by (6.2). Then  $L_y$  is said to be separated in  $L^2(0, \infty)$  if and only if for  $f$  in  $D(q)$ ,

- (i)  $q$  is locally square integrable on  $0 \leq x < \infty$ , and
- (ii)  $qf$  is in  $L^2(0, \infty)$ .

Note that  $L_y$  is separated in  $L^2(0, \infty)$  if in addition to  $Lf$  being in  $L^2(0, \infty)$ , each of the terms of  $Lf$  is also in  $L^2(0, \infty)$ . Several papers have appeared concerning these concepts. An expository paper presenting these concepts in a unified manner and connecting these with known limit-point criteria would be of interest.

The bibliography given here is not restricted to those works upon which this paper depends. One of the goals in undertaking the research for this paper has been to locate those papers and books that deal in a general way with eigenfunction expansion theorems and the deficiency index problem. With this bibliography, it is hoped that anyone wishing to consider some aspect of the topics presented here or some of the generalizations to the concepts mentioned above will find a comprehensive listing of appropriate sources with which to begin a study.

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2  
VITA

David Hartman Taylor

Candidate for the Degree of

Doctor of Education

Thesis: THE LIMIT-POINT AND LIMIT-CIRCLE CASES OF SECOND AND FOURTH  
ORDER DIFFERENTIAL EQUATIONS

Major Field: Higher Education

Biographical:

Personal Data: Born in Santa Ana, California, September 14, 1940,  
the son of William Alva and Lorene Taylor; married to Rosalee  
Joy Tugwell, December 26, 1966.

Education: Attended elementary school in Poteau, Oklahoma,  
Hallett, Oklahoma, Midwest City, Oklahoma, Uvalde, Texas, and  
Wharton, Texas; graduated from Terlton High School, Terlton,  
Oklahoma in May, 1958; attended Oklahoma State University,  
June, 1958-January, 1961; received the Bachelor of Science  
degree with a major in mathematics from Southwestern Oklahoma  
State University, Weatherford, Oklahoma in May, 1966; received  
the Master of Arts degree from the University of Missouri-  
Columbia, Columbia, Missouri in June, 1968; entered Oklahoma  
State University in September, 1971, and completed require-  
ments for the Doctor of Education degree in May, 1975.

Professional Experience: Served with the United States Army, 1961-  
1964; worked as a teaching assistant at Southwestern Oklahoma  
State University, Weatherford, Oklahoma, 1965-1966; graduate  
assistant at the University of Missouri-Columbia, Columbia,  
Missouri, 1966-1968; instructor of mathematics at Southwestern  
Oklahoma State University, Weatherford, Oklahoma, 1968-1971;  
graduate assistant at Oklahoma State University, 1971-1974;  
presently employed in the Mathematics Department at South-  
western Oklahoma State University, 1974.

Professional Organizations: The Mathematical Association of  
America.